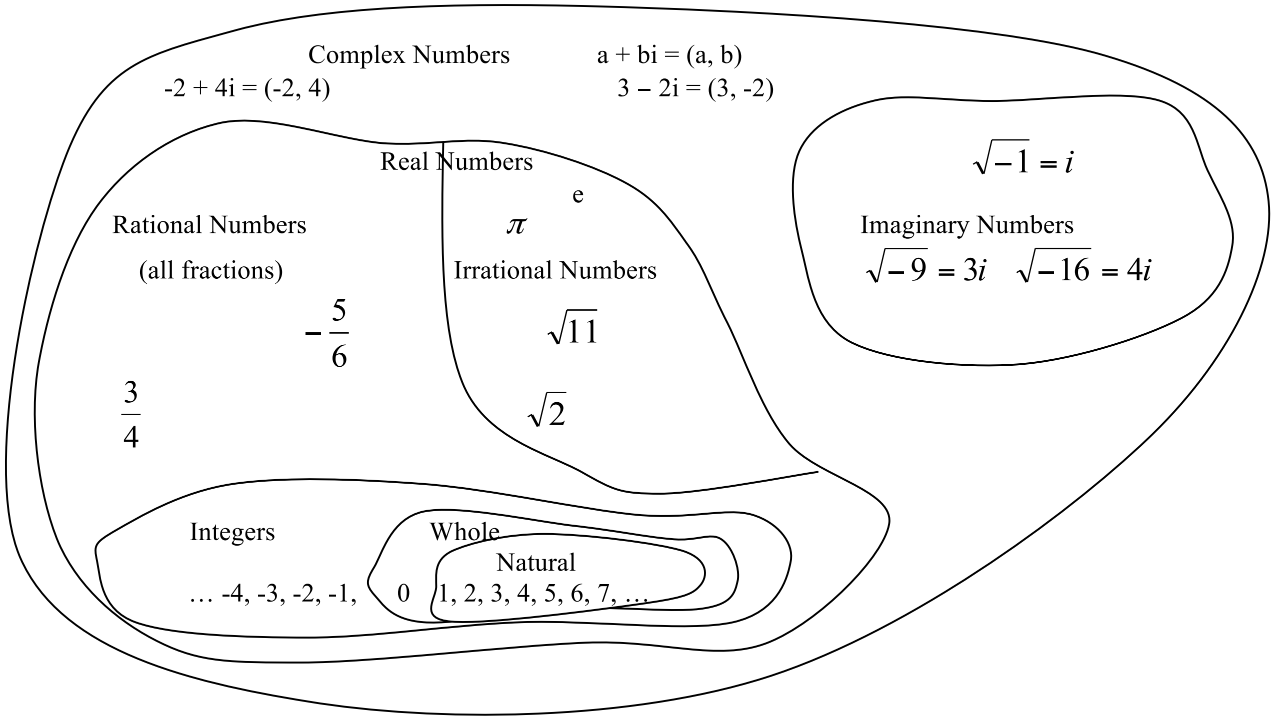
Math 112 – Lecture I – Review of Algebra and Trig, plus

A) Complex Numbers –

1) Venn Diagram



2) Operations with Complex Numbers, particularly division

a) All other operations: add, subtract, and multiply are treated just like binomials (they must be written in term form and for multiplication remember that $i^2 = -1$) ; i.e. $(3 – 4i)(-2 + 5i) = -6 + 15i + 8i – 20i^2 = 14 + 23i$

b) Division **by hand** requires the term form as well, plus **Multiplication by ONE**; i.e.

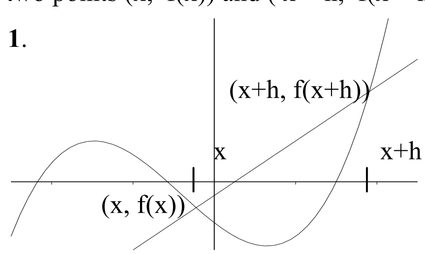
$$\frac{3-4i}{-2+5i}\cdot 1 = \frac{3-4i}{-2+5i}\cdot\frac{-2-5i}{-2-5i} = \frac{-6-15i+8i+20i^2}{4+10i-10i-25i^2} = \frac{-26-7i}{29} = \frac{-26}{29} - \frac{7}{29}i$$

Multiplying by ONE is the **only thing you can do to a fraction** – we will use this almost daily this semester.

B) Functions – Most of our work will be with functions this semester and so we must understand function notation: y = f(x) and what it means to write f(-2), or f(3-2x), etc. [it is just a matter of substituting '-2' or '3-2x' in for the x in f(x)]

B1) The Special Slope formula is the slope of the line containing the two points (x, f(x)) and (x + h, f(x + h)):

$$m = \frac{rise}{run} = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$$ **This is Lesson Plan 1**.



Example 1: What is the special slope for $f(x) = 3x – 4x^2$?

$f(x+h) = 3(x+h) – 4(x+h)^2 = 3x + 3h – 4(x^2+2xh+h^2)$

$= 3x + 3h – 4x^2 – 8xh – 4h^2$ so

$$\mathbf{m} = \frac{(3x+3h-4x^2-8xh-4h^2)-(3x-4x^2)}{h}$$

$$= \frac{3x+3h-4x^2-8xh-4h^2-3x+4x^2}{h} = \frac{3h-8xh-4h^2}{h} = \mathbf{3 - 8x - 4h}$$

Example 2: What is the special slope for $f(x) = \frac{1-2x}{4x-6}$? $\quad f(x+h) = \frac{1-2(x+h)}{4(x+h)-6} = \frac{1-2x-2h}{4x+4h-6}$ so

$$m = \frac{\frac{1-2x-2h}{4x+4h-6} - \frac{1-2x}{4x-6}}{h}\cdot 1 = \frac{\frac{1-2x-2h}{4x+4h-6} - \frac{1-2x}{4x-6}}{h}\cdot\frac{(4x+4h-6)(4x-6)}{(4x+4h-6)(4x-6)} =$$

$= \dfrac{(1-2x-2h)(4x-6)-(1-2x)(4x+4h-6)}{h(4x+4h-6)(4x-6)} = \dfrac{4x-8x^2-8xh-6+12x+12h-4x+8x^2-4h+8xh+6-12x}{h(4x+4h-6)(4x-6)}$ which simplifies to: $m = \dfrac{8h}{h(4x+4h-6)(4x-6)} = \dfrac{8}{(4x+4h-6)(4x-6)} = \dfrac{8}{2(2x+2h-3)\cdot 2(2x-3)} = \dfrac{2}{(2x+2h-3)(2x-3)}$ Notice both of the 2's in the denominator canceled with 8 to get 2 in the numerator. Also notice that in both examples the factor h in the denominator canceled out.

B2) Graphing Rational Functions requires the following 5 steps:

1) **Factor** the numerator and the denominator **and reduce** if possible. 'Poly' can be used if desired, if the same root is in the numerator and the denominator canceling them is the same as reducing.
2) The **Real roots** of the **numerator** are the only **x-intercepts**, graph them (remember multiplicity).
3) The **Real roots** of the **denominator** are **vertical asymptotes**, graph them as vertical dotted lines, if they occur an odd number of times call the vertical asymptote **ODD** and the graph goes to opposite ends on each side, if an even number of times call it **EVEN** and the graph goes to the same end on opposite sides.
4) **Compare** the **degree** of the numerator, N, with the **degree** of the denominator, D:

If **N < D**, there is a **horizontal asymptote** at **y = 0**, since for large values of x the denominator would be much larger than the numerator. To see if the graph is above the line y = 0 (x-axis) or below on the far right, consider if y is positive for a large x or negative (dividing the leading coefficients will determine this).

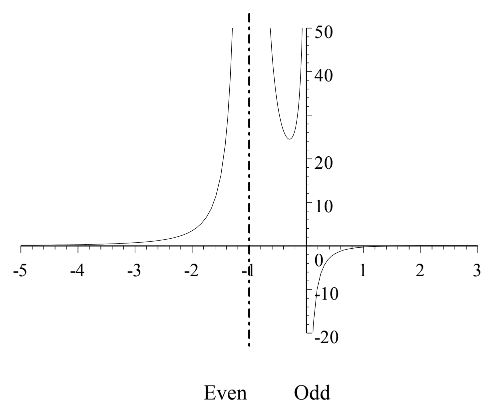
If **N = D**, there is a **horizontal asymptote** at $\mathbf{y = \frac{a}{b}}$**,** where **a and b** are the **leading coefficients** respectively.

If **N > D**, there is an **oblique asymptote** at **y = Q(x)**, Q(x) is the quotient obtained from **long division**.

5) To validate your graph select another point or two (it is always best to begin graphing from the right).

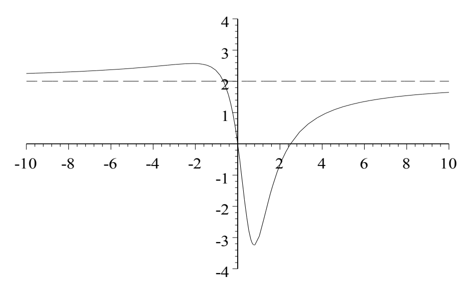
Example 1: $y = \dfrac{2x^2 - 3x}{x^2 + 2x^3 + x^4}$; 1) factoring gives $y = \dfrac{x(2x-3)}{x\ x(1+x)(1+x)}$ and reducing gives $y = \dfrac{(2x-3)}{x(1+x)(1+x)}$

2) The only real root of the numerator is 1.5, so it is the only x-intercept.
3) The real roots of the denominator are 0, and -1 twice, so there is an ODD vertical asymptote at 0 and an EVEN one at -1.
4) The degree on top is 2 and on the bottom is 4, N<D, so a horizontal asymptote exists at y = 0, and for a large x value it is positive so the graph is above the x-axis on the right of 1.5.
5) Other points could be chosen but none are necessary.



Example 2: $y = \dfrac{2x^3 - 3x^2 - 5x}{x^3 + 1}$; 1) using 'poly' the roots are: $\dfrac{0 \quad 2.5 \quad \cancel{-1}}{\cancel{-1} \quad (.5,.87)(.5,-.87)}$, cancel the –1

2) The real roots in numerator are: 0 and 2.5, so these are the only x-intercepts.
3) There are no real roots left in the denominator, so there are no vertical asymptotes.
4) The degrees are the same so N = D, therefore the horizontal asymptote is $y = \dfrac{2}{1} = 2$ (leading coefficient over leading coefficient).
5) The graph starts at the horizontal asymptote on the far right so no other points are necessary.

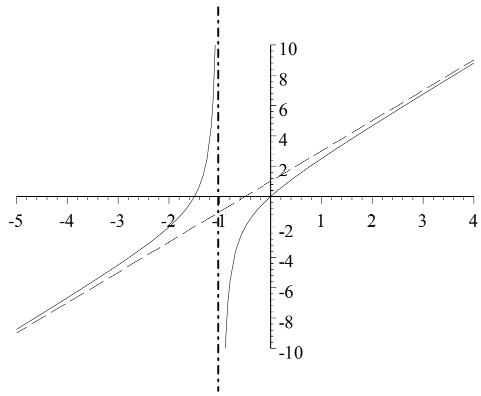


Example 3: $y = \dfrac{2x^3 + x^2 - 3x}{x^2 - 1}$; 1) factoring gives $y = \dfrac{x(2x+3)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$

2) The real roots in numerator are: 0 and –1.5, so they are the only x-intercepts
3) The only real root left in the denominator is –1, so the vertical asymptote at –1 is ODD.
4) The degree on top is 3, on bottom is 2 so, N > D. Therefore there is an oblique asymptote at y = Q(x), or in this case at

y = 2x + 1, since:

$$\begin{array}{r|l} & 2x+1 \\ \hline x^2-1 & 2x^3 + x^2 - 3x \\ & -2x^3 \quad + 2x \\ \hline & x^2 - x \\ & -x^2 \quad + 1 \\ \hline \end{array}$$



B3) Composition of functions and Inverses of functions.

1) **Composition** of functions is nothing more than the function of a function [for compositions y is always replaced with f(x) or g(x)]. Example, suppose $f(x) = 3x - 2x^2$ and $g(x) = 4 - 3x$, then **f composite g** is:

$f \circ g = f(g(x)) = f(4-3x) = 3(4-3x) - 2(4-3x)^2 = 12 - 9x - 2(16 - 24x + 9x^2) = 12 - 9x - 32 + 48x - 18x^2$ $= -20 + 39x - 18x^2$ and **g composite f** is: $g \circ f = g(f(x)) = g(3x - 2x^2) = 4 - 3(3x - 2x^2) = 4 - 9x + 6x^2$.

2) **Inverses** exist only for functions that are **one-to-one**, which means that not only is there one and only one y for each x but there must also be one and only one x for each y (functions must pass a horizontal line test as well as the vertical line test). A function that does have an inverse is $h(x) = \dfrac{3x+1}{2-x}$, it is one-to-one since it is a linear function divided by another linear function. You **find the inverse of a function, h(x), by**:

-Switching x and y and

-Solving for y

The result you get for **y is the inverse of h(x), written $h^{-1}(x)$.** Therefore $\mathbf{h^{-1}(x)} = \dfrac{2x-1}{3+x}$.

3) Inverse Function Rule combines composition of functions with Inverses: $f \circ f^{-1} = x = f^{-1} \circ f$

Example using h(x) above: $h \circ h^{-1} = \dfrac{3\dfrac{2x-1}{3+x} + 1}{2 - \dfrac{2x-1}{3+x}} \cdot \dfrac{3+x}{3+x} = \dfrac{3(2x-1) + 1(3+x)}{2(3+x) - (2x-1)} = \dfrac{6x - 3 + 3 + x}{6 + 2x - 2x + 1} = \dfrac{7x}{7} = x$.

You would also get x if you worked out $h^{-1} \circ h$, try it as a homework problem.

C) Logarithms – a new country with new laws, rules and standards, pay attention to these Rules:

a) **Definition**: **If $y = b^x$ then $x = \log_b(y)$**; examples: $6^{2-x} = 11$ ⟷ $2 - x = \log_6(11)$; $\log_2(3x - 4) = 4$ ⟷ $2^4 = 3x - 4$; $10^x = 15$ ⟷ $x = \mathbf{\log_{10}}(15)$, however when the base is 10 we do not write it, so $x = \mathbf{\log}(15)$; similarly when the base is 'e', like in $\mathbf{\log_e}(x) = 4$, we write $\mathbf{ln}(x) = 4$ instead, so if $e^{2x} = 5$ ⟷ $\ln(5) = 2x$; The double arrow means '**by definition**'. Homework problem: $y = 2^x$ and $y = \log_2 x$ are inverses; prove it.

b) **Identities** (4 and they come from the definition applied to: $b^0 = 1$; $b^1 = b$; $b^n = b^n$; and $\log_b(x) = \log_b(x)$):

i) $\log_b(1) = 0$ examples: $\log_2(1) = 0$; $\log(1) = 0$; $\ln(1) = 0$; $\log_8(1) = 0$

ii) $\log_b(b) = b$ examples: $\log_2(2) = 1$; $\log(10) = 1$; $\ln(e) = 1$; $\log_6(6) = 1$

iii) $\log_b(b^n) = n$ examples: $\log_2(8) = 3$; $\log(100) = 2$; $\ln(e^{-1}) = -1$; $\log_{25}(5) = .5$

iv) $b^{\log_b(x)} = x$ examples: $2^{\log_2(x)} = x$; $10^{\log(4)} = 4$; $e^{\ln(x)} = x$; $3^{\log_3(2x+1)} = 2x + 1$

c) Laws (4 also)

i) $\log_b(xy) = \log_b(x) + \log_b(y)$ examples: $\log_2(3x) = \log_2(3) + \log_2(x)$; $\ln(12) = \ln(2) + \ln(6)$.

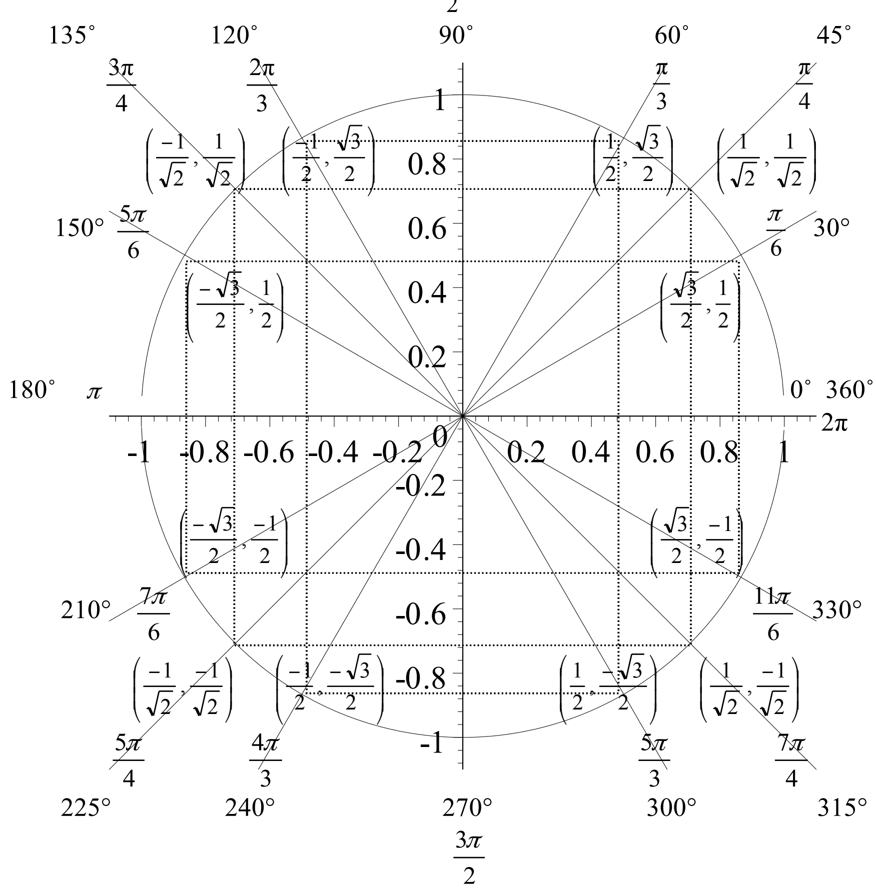
ii) $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$ examples: $\log\left(\dfrac{5}{12}\right) = \log(5) - \log(12)$; $\log_3\left(\dfrac{x}{6}\right) = \log_3(x) - \log_3(6)$.

iii) $\log_b(x^n) = n\log_b(x)$ examples: $\ln(x^4) = 4\ln(x)$; $2\log_5(y) = \log_5(y^2)$; $3\log_2(5) = \log_2(125)$.

iv) $\log_b(x) = \dfrac{\log_a(x)}{\log_a(b)}$ examples: $\log_4(6) = \dfrac{\log(6)}{\log(4)}$; $\log_8(4) = \dfrac{\log_2(4)}{\log_2(8)} = \dfrac{2}{3}$; $\log_5(20) = \dfrac{\ln(20)}{\ln(5)}$

D) Trigonometry

1) The Unit Circle with 16 points and 17 angles + infinity.



2) There are six trig functions and they are defined using the x and y from the points on the unit circle as follows (with **reciprocal functions paired**, i.e. sin θ is the reciprocal of csc θ, etc.):

a) $\sin\theta = y$ f) $\csc\theta = \frac{1}{y}$ Example for $\frac{2\pi}{3}$: a) $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ f) $\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}}$

b) $\cos\theta = x$ e) $\sec\theta = \frac{1}{x}$ b) $\cos\frac{2\pi}{3} = \frac{-1}{2}$ e) $\sec\frac{2\pi}{3} = -2$

c) $\tan\theta = \frac{y}{x}$ d) $\cot\theta = \frac{x}{y}$ c) $\tan\frac{2\pi}{3} = -\sqrt{3}$ d) $\cot\frac{2\pi}{3} = \frac{-1}{\sqrt{3}}$

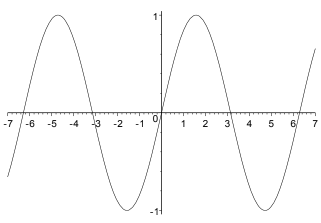
3) Using the definitions of part two and the Unit Circle of part one it is possible to construct the following table (this is called the **Restricted Table**): Notice each function is one-to-one:

θ	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Sin θ	-1	$-\frac{\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1				
Cos θ					1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1
Tan θ	und	$-\sqrt{3}$	-1	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und				
Cot θ					und	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{-1}{\sqrt{3}}$	-1	$-\sqrt{3}$	und
Sec θ					1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	-1
Csc θ	-1	$\frac{-2}{\sqrt{3}}$	$-\sqrt{2}$	-2	und	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1				

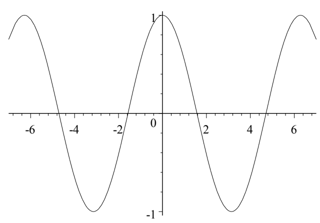
We could have filled in every box but doing so gives functions that are not one-to-one. The purpose of this table is to change the six trig functions to **one-to-one functions** so they can have inverses. For **one** of your **homework** problems prepare a table of all 17 angles in the Unit Circle (use that table to graph the functions in part 4).

4) From the homework problem table it is possible to graph each of the six trig functions by letting the x values be the angles from the top row of the table and the numbers in the function row be the y values respectively. The graphs follow (for the graphs all the fractions from the table are changed to decimal numbers, including the angles):

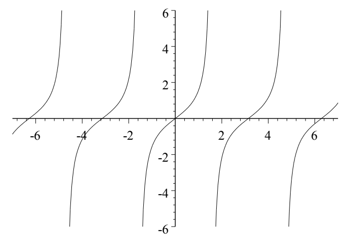
y = sin x



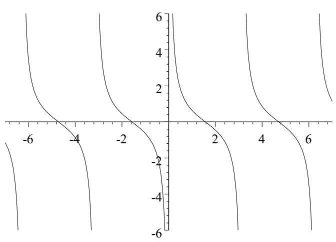
y = cos x



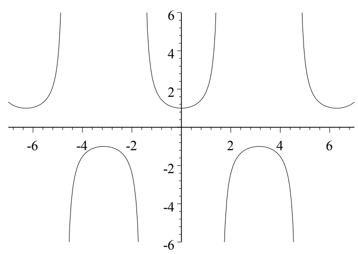
y = tan x



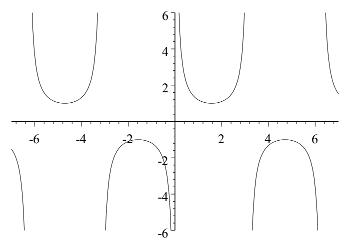
y = cot x



y = sec x



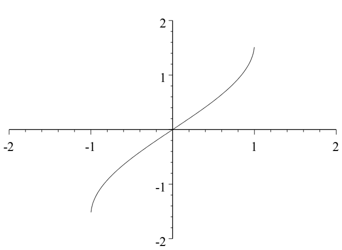
y = csc x



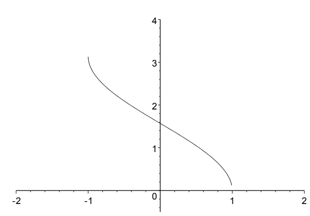
Notice these graphs repeat, the repeating **period** for all functions except tan x and cot x is **2π** (or in decimals 6.28). The repeating **period** for **tan x and cot x** is just ***π*** (3.14). Because these graphs repeat **they are not one-to-one** and therefore as they stand they do not have inverses. Also notice 'und' always translates into a vertical asymptote.

5) Using the Restricted table in part 3 and switching the x and y (meaning the y values now are the angles and the x-values are opposite the function name) we are able to graph the Inverse Trig functions given below:

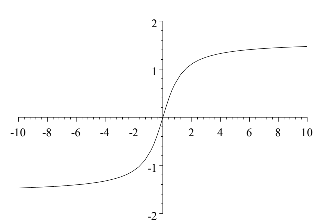
$y = \sin^{-1} x$ (y = arcsin x)



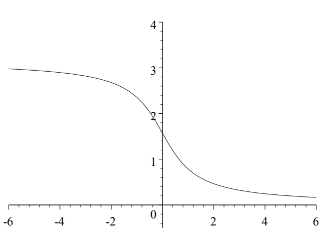
$y = \cos^{-1} x$



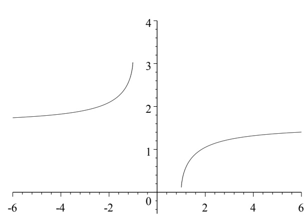
$y = \tan^{-1} x$



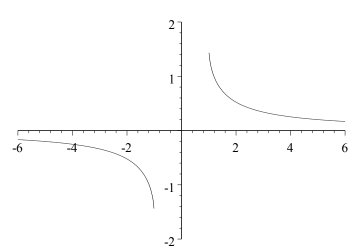
$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \csc^{-1} x$



Notice these functions are one-to-one, they do not repeat, for every x-value there is only one y-value.

6) The Magnificent Seven Identities – all the Trig Identities needed for Calculus:

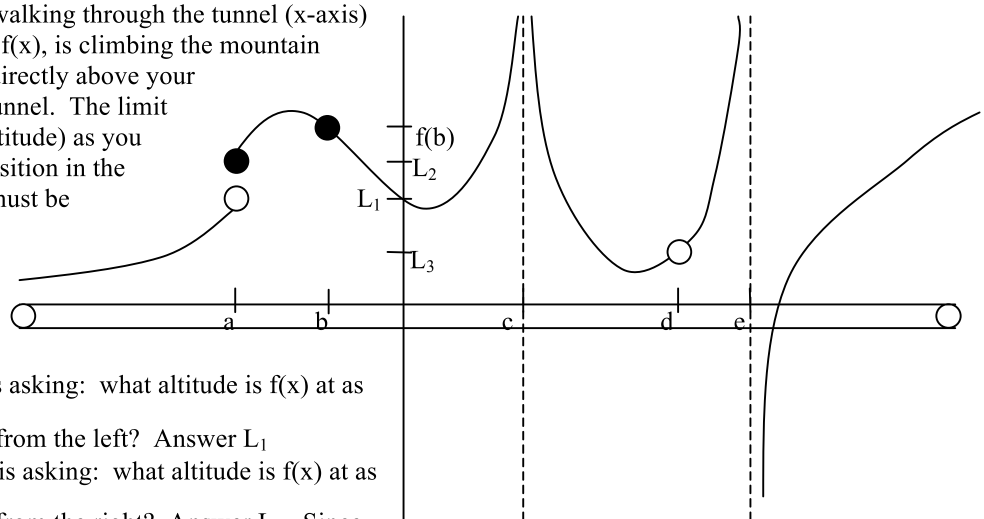
a) $\mathbf{\cos^2 x + \sin^2 x = 1}$ or by dividing by $\cos^2 x$: $\mathbf{1 + \tan^2 x = \sec^2 x}$ or by $\sin^2 x$: $\mathbf{\cot^2 x + 1 = \csc^2 x}$

b) **cos(x + y) = cos x cos y – sin x sin y** called the sum formula for cosines.

c) $\mathbf{\sin(x + y) = \sin x \cos y + \sin y \cos x}$ called the sum formula for sines.
d) $\mathbf{\cos(2x) = \cos^2 x - \sin^2 x}$ or using (a) and substituting: $\mathbf{\cos(2x) = 1 - 2\sin^2 x}$ and $\mathbf{\cos(2x) = 2\cos^2 x - 1}$.
e) $\mathbf{\sin(2x) = 2 \sin x \cos x}$. Notice d and e come from b and c by letting y equal x.
f) $\mathbf{\sin^2 x} = \dfrac{1 - \cos(2x)}{2}$. Notice this identity comes from the second form of d, solving for $\sin^2 x$.

E) Limits – one of the three key ideas for Calculus
1) Definitions:
a) Intuitive definition for the limit of f(x) as x approaches a, written: $\lim_{x \to a} f(x) = L$

The **Limit, L**, is a **unique Altitude** (y-value)
Imagine you are walking through the tunnel (x-axis) and your friend, f(x), is climbing the mountain always staying directly above your position in the tunnel. The limit (your friend's altitude) as you approach any position in the tunnel is what must be determined.



$\lim_{x \to a^-} f(x) = L_1$ is asking: what altitude is f(x) at as x approaches a from the left? Answer L_1
$\lim_{x \to a^{+-}} f(x) = L_2$ is asking: what altitude is f(x) at as x approaches a from the right? Answer L_2. Since these are different altitudes we conclude that $\lim_{x \to a} f(x) = \text{undefined}$. On the otherhand $\lim_{x \to b} f(x) = f(b)$; when the limit is the y-value at a point, the function is said to be **continuous** at that point. Other limits of interest are: $\lim_{x \to c} f(x) = \infty$; $\lim_{x \to d} f(x) = L_3$; $\lim_{x \to e} f(x) = \text{und}$, since the graph is going up on the right but down on the left, which is obviously different altitudes.

b) Math Definition: $\lim_{x \to a} f(x) = L$ providing for every ε>0 there exists a δ>0 such that |f(x) – L|<ε whenever 0<|x-a|<δ.

Example: If $f(x) = 4 - 3x - x^2$ and if a = -2, then $\lim_{x \to -2} 4 - 3x - x^2 = 6$ providing for every ε>0 there exists a δ>0 such that

$|4 - 3x - x^2 - 6| < \varepsilon$ whenever $0<|x+2|<\delta$, or simplified: $|-2 - 3x - x^2| < \varepsilon$ whenever $0<|x+2|<\delta$.

2. Limits can be evaluated three ways (a fourth way will be presented in a later lecture), each limit **must** be done **two** ways:
a) Graphical – graph the function and use the intuitive definition – all functions can be graphed and the only time the limit is undefined is at cliffs or odd asymptotes. This method gives only an estimate, however.
b) Numerical – create a table of values close to 'a' on both sides and find the corresponding y-values. This gives a precise limit to any degree of accuracy desired. If 'a' was 2, the values close to 2 could be: 1.9, 1.99, 1.999, 2.1, 2.01, 2.001.
c) Symbolical – factor the algebra expression for f(x) and reduce it, then substitute in the values for 'a', the answer is the limit. This method only works when the function can be reduced.

Example 1. $\lim_{x \to 2} \dfrac{x^2 - 4}{2 - 3x + x^2}$ since f(x) can be factored and reduced: $\dfrac{(x-2)(x+2)}{(x-2)(x-1)} = \dfrac{(x+2)}{(x-1)}$ and 2 plugged in gives 4, the limit is 4. Graphing the original rational function gives a hole in the graph at (2,4), but the limit exists and is 4.