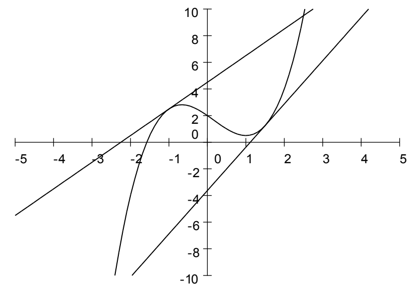
Math 112 – Lecture II – Calculus is the Limit, Derivatives too:

A) The Two definitions for the derivative:



1) The Intuitive Definition: f'(x) or $\frac{dy}{dx}$ is the slope of the tangent line at a point (x, f(x)). So the derivative of f(x) when x is -1 is the slope of the tangent line where x is -1 (about 2), and the derivative of f(x) at x = 1.5 is about 3. What would your guess be when x = 1? ____; when x = 0? ____

2) The Math definition of f'(x) or $\frac{dy}{dx}$ is: $f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

Note: As h approaches 0, the point (x+h, f(x+h)) moves along the function toward the point (x,f(x)), (see the graph in lecture I) and so f'(x) or $\frac{dy}{dx}$ becomes the slope of the tangent line at the single point (x, f(x)).

B) The Rules for derivatives are obtained from the definition applied to certain functions:

1) The constant rule: **(c)' = 0** since for f(x) = 5, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{5-5}{h} = 0$

2) The power rule: $\mathbf{(x^n)' = n\,x^{n-1}}$ example since for $f(x) = x^3$,

$$f'(x) = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h} = \lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-x^3}{h} = \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}3x^2+3xh+h^2 = 3x^2$$

3) The sum rule: **(f(x) + g(x))' = f'(x) + g'(x)**

The proof goes as follows:

$$\textbf{(f(x) + g(x))'} = \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h} = \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}$$ but this can be rewritten as: $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$ which is nothing more than **f'(x) + g'(x)**.

4) The product rule: **(f(x)·g(x))' = g(x)·f'(x) + f(x)·g'(x)**

The proof goes as follows:

$$\textbf{(f(x)·g(x))'} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-\mathbf{g(x+h)f(x)}+\mathbf{g(x+h)f(x)}-f(x)g(x)}{h}$$; Notice the same extra term was subtracted and then added between the original two terms. Now we split it in half and factor:

$= \lim_{h\to 0}\frac{g(x+h)[f(x+h)-f(x)]}{h}+\lim_{h\to 0}\frac{f(x)[g(x+h)-g(x)]}{h}$, and because of the rules for limits we can set up 4 limits:

$= \lim_{h\to 0}g(x+h)\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}f(x)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$, which simplify to: **g(x)·f'(x) + f(x)·g'(x)**

5) The quotient rule: $\left(\frac{\mathbf{f(x)}}{\mathbf{g(x)}}\right)' = \frac{\mathbf{g(x)f'(x)-f(x)g'(x)}}{\mathbf{[g(x)]^2}}$

The proof goes as follows: $\left(\frac{\mathbf{f(x)}}{\mathbf{g(x)}}\right)' = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)}-\frac{f(x)}{g(x)}}{h}\cdot\frac{g(x+h)g(x)}{g(x+h)g(x)} = \lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x+h)}{h\cdot g(x+h)g(x)}$ but now we can again subtract and add the same extra term, f(x)g(x), in the numerator and divide things up as follows:

$$= \lim_{h\to 0}\frac{f(x+h)g(x)-\mathbf{f(x)g(x)}-f(x)g(x+h)+\mathbf{f(x)g(x)}}{h\cdot g(x+h)g(x)} = \lim_{h\to 0}\left(g(x)\cdot\frac{f(x+h)-f(x)}{h}-f(x)\cdot\frac{g(x+h)-g(x)}{h}\right)\cdot\frac{1}{g(x+h)g(x)}$$

And because of the rules for limits we can take the limit of each factor separately inside the parenthesis and the last factor too giving us: [g(x)·f'(x)-f(x)·g'(x)] over g(x)·g(x) or $\dfrac{\mathbf{g(x)f'(x)-f(x)g'(x)}}{\mathbf{[g(x)]^2}}$

An easy way to remember this rule is: $\dfrac{\text{ho dhi} - \text{hi dho}}{\text{ho ho}}$ where the numerator is 'hi' and the denominator is 'ho', the derivatives just require a 'd' in front.

6) The six trig function derivatives:

a) (sin x)' = cos x

To prove the sin x and cos x rules the following two limits are needed: $\lim_{h\to 0}\dfrac{\sin(h)}{h} = 1$ and $\lim_{h\to 0}\dfrac{\cos(h)-1}{h} = 0$ you should be able to verify both limits at least two ways – do this as homework.

Proof for sin x: (sin x)' = $\lim_{h\to 0}\dfrac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\dfrac{\sin h\cos x+\cos h\sin x-\sin x}{h}$ and factoring out a sin x in the last terms and simplifying gives: $\lim_{h\to 0}\left(\dfrac{\sin h}{h}\cdot\cos x+\sin x\cdot\dfrac{\cos h-1}{h}\right)$ and by making 4 separate limits we get: 1·cos x + sin x ·0 = cos x

b) (cos x)' = -sin x

Proof for cos x: (cos x)' = $\lim_{h\to 0}\dfrac{\cos(x+h)-\cos x}{h} = \lim_{h\to 0}\dfrac{\cos x\cos h-\sin x\sin h-\cos x}{h}$, which is equal to: $\lim_{h\to 0}\dfrac{\cos x\cos h-\cos x-\sin x\sin h}{h} = \lim_{h\to 0}\dfrac{\cos x\,(\cos h-1)}{h} - \lim_{h\to 0}\sin x\,\dfrac{\sin h}{h}$, and the 4 limits give: cos x · 0 – sin x · 1 = -sin x

c) (tan x)' = sec²x

Proof for tan x (as well as the remaining trig rules) uses the **quotient rule** rather than the definition:

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{\cos x\cdot\cos x-\sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

d) (cot x)' = -csc²x

Proof for cot x: $$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{\sin x\cdot -\sin x-\cos x\cdot\cos x)}{\sin^2 x} = \frac{-\sin^2 x-\cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x$$

e) (sec x)' = sec x · tan x

Proof for sec x: $$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \frac{\cos x\cdot 0-1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x\cdot\tan x$$

f) (csc x)' = -csc x · cot x

Proof for csc x: $$(\csc x)' = \left(\frac{1}{\sin x}\right)' = \frac{\sin x\cdot 0-1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = \frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\cdot\cot x$$

7) The exponential and log rules:

a) **$(e^x)' = e^x$** Proof: $$\lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h-e^x}{h} = \lim_{h\to 0} e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x\cdot 1 = e^x$$

Homework problem: Show that $\lim_{h\to 0}\dfrac{e^h-1}{h} = 1$ and show it two ways.

b) $(a^x)' = a^x \cdot \ln a$

Proof: $(5^x)' = \lim_{h\to 0}\dfrac{5^{x+h}-5^x}{h} = \lim_{h\to 0}\dfrac{5^x 5^h-5^x}{h} = \lim_{h\to 0}5^x \lim_{h\to 0}\dfrac{5^h-1}{h} = 5^x\cdot 1.609$ but 1.609 = ln 5, so the derivative of 5^x does equal $5^x \cdot \ln 5$ and it turns out that 5 could be replaced with any constant.

c) **(ln x)'** = $\frac{\mathbf{1}}{\mathbf{x}}$ The proof for this rule will be given after the chain rule and implicit differentiation.

8) The Chain Rule or the derivative of a composition of functions: $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ where z represents the function inside another function.

Examples: $(\sin x^2)' = (\sin z)' \cdot (x^2)' = \cos z \cdot 2x = 2x\cos(x^2)$

$(\cos^3 x)' = (z^3)' \cdot (\cos x)' = 3z^2 \cdot (-\sin x) = 3\cos^2 x \cdot (-\sin x) = -3\sin x \cos^2 x$

$(3^{\tan x})' = (3^z)' \cdot (\tan x)' = 3^z \ln 3 \cdot \sec^2 x = 3^z \ln 3 \sec^2 x$

$(\ln(\csc x))' = (\ln(z))' (\csc x)' = \frac{1}{z} \cdot (-\csc x \cot x) = \frac{-\csc x \cot x}{\csc x} = -\cot x$

Note: This rule can be applied to more than two links: $\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dz} \cdot \frac{dz}{dx}$

Example: $(\cot^2 x^3)' = 2\cot x^3 \cdot (-\csc^2 x^3) \cdot 3x^2 = -6x^2 \cot x^3 \csc^2 x^3$ can you see the three derivative rules used?

9) Impicit Differentiation – finding derivatives of functions when you are not able to solve for y. When it is not possible to solve for y in a given expression and you are asked to find y' or $\frac{dy}{dx}$, you must apply the **chain rule** whenever the derivative of y is required.

Example: Find $\frac{dy}{dx}$ given $\sin(xy) = y^2 + x^2$. Notice it is impossible to solve for y, so we must take the derivative of both sides with respect to x using the chain rule and since we have x times y inside the sine function the product rule will also be used: $(\sin(xy))' = \cos(xy)\,(y \cdot 1 + x\frac{dy}{dx})$ and $(y^2 + x^2)' = 2y\frac{dy}{dx} + 2x$, setting both sides equal gives: $\cos(xy)\,(y + x\frac{dy}{dx}) = 2y\frac{dy}{dx} + 2x$. Removing parenthesis on the left gives: $y\cos(xy) + x\cos(xy)\frac{dy}{dx} = 2y\frac{dy}{dx} + 2x$ and since we must solve for $\frac{dy}{dx}$ terms can be moved to get: $x\cos(xy)\frac{dy}{dx} - 2y\frac{dy}{dx} = 2x - y\cos(xy)$. Now factor the left side getting: $(x\cos(xy) - 2y)\frac{dy}{dx} = 2x - y\cos(xy)$. Therefore: $\frac{dy}{dx} = \frac{2x - y\cos(xy)}{x\cos(xy) - 2y}$.

Note: The proof for ln(x) is as follows: If $y = \ln(x)$ then by the log definition $e^y = x$, now take the derivative of both sides of the equals: $e^y \frac{dy}{dx} = 1$ and solving for $\frac{dy}{dx}$ gives $\frac{dy}{dx} = \frac{1}{e^y}$, but $e^y = x$ so $(\ln(x))' = \frac{1}{x}$.

10) The 6 Inverse Trig functions, and their derivatives:

a) $\left(\sin^{-1} x\right)' = \frac{\mathbf{1}}{\sqrt{\mathbf{1} - x^{\mathbf{2}}}}$

b) $\left(\cos^{-1} x\right)' = \frac{\mathbf{-1}}{\sqrt{\mathbf{1} - x^{\mathbf{2}}}}$

c) $\left(\tan^{-1} x\right)' = \frac{\mathbf{1}}{\mathbf{1} + x^{\mathbf{2}}}$

d) $\left(\cot^{-1} x\right)' = \frac{\mathbf{-1}}{\mathbf{1} + x^{\mathbf{2}}}$

e) $\left(\sec^{-1} x\right)' = \frac{\mathbf{1}}{x\sqrt{x^{\mathbf{2}} - \mathbf{1}}}$

f) $\left(\csc^{-1} x\right)' = \frac{\mathbf{-1}}{x\sqrt{x^{\mathbf{2}} - \mathbf{1}}}$

Note: The proofs for the Inverse Rules go as follows: for the $(\cos^{-1} x)$, let $y = \cos^{-1} x$, by taking the cos of both sides we get, $\cos y = x$ and taking the derivative of both sides with respect to x gives, $-\sin y \frac{dy}{dx} = 1$, solving for $\frac{dy}{dx}$, $\frac{dy}{dx} = \frac{-1}{\sin y}$ but from identity one, $\sin y = \sqrt{1 - \cos^2 y}$ and since $\cos y = x$, $\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$. All the others are done in a similar way using different forms of Identity One. Try the rest as **homework** problems.