Math 112 – Lecture III Applications of the Derivative

A) Related Rate Games: All rates are written $\frac{d(\text{something})}{dt}$, and represent how something changes with respect to time. All these applications can be solved using the following steps:

1) Write down all rates. There will always be one rate that is unknown, all others will be known.
2) Write one equation having only the variables in the numerators of the rates.
3) Find the derivative with respect to time.
4) Plug in the moment of time (every variable must be known at the moment of time except the rate you must find), and solve for the desired rate.

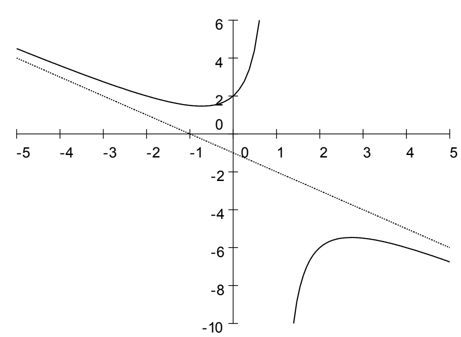
Example: How fast is the volume of a pile of wheat changing if the wheat is falling into a cone shaped pile with the height rising at 1 foot per minute and the radius expanding at 2 feet per minute when the height is 4 feet and the radius is 6 feet?

1) The rates are $\frac{dV}{dt} = ?$ $\frac{dh}{dt} = 1 \text{ ft per min}$ and $\frac{dr}{dt} = 2 \text{ ft per minute}$
2) Our equation having only the variables V, h and r; it is: $V = \frac{1}{3}\pi r^2 h$ for the volume of a cone
3) Take the derivative with respect to time: $\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right)$, Notice the product rule was needed
4) Plug in the moment of time (when height was 4 and the radius was 6): $\frac{dV}{dt} = \frac{\pi}{3}\left(2 \cdot 6 \cdot 4 \cdot 2 + 6^2 \cdot 1\right) = 138.23\frac{ft^2}{min}$

B) Optimization applications involve graphing and games.

1) Graphing functions requires understanding optimization and therefore derivatives, the steps are:
   a) Use past graphing techniques if you can apply them fast
   b) Find all **critical values** (x values only) by taking the **derivative**, f'(x), and finding **all real roots** of it's **numerator** and it's **denominator**. Find the corresponding y-value for each critical value by substituting them into f(x), thus getting the **Critical Points**. Note: **These points are where the slope of the tangent line is zero** or undefined.
   c) Find all **inflection values** by taking the **second derivative**, f''(x), and finding **all real roots** of it's **numerator** and it's **denominator**. Find the corresponding y-value for each inflection value by substituting them into f(x), thus getting the **Inflection Points**. Note: **These points are where the concavity changes**.
   d) The Second Derivative test for extrema, plug all **critical values** into **f''(x)**, if:
      i) f''(cv) = a **positive** number, the corresponding critical point is a **minimum point**.
      ii) f''(cv) = a **negative** number, the corresponding critical point is a **maximum** point.
      iii) f''(cv) = **zero**, the critical point is also an inflection point.
      iv) f''(cv) = **undefined** you cannot make any conclusion about the critical point.

   Note: The **second derivative** test works because the second derivative at any x-value **tells the concavity** of the graph at that x-value. This is the **intuitive definition for f''(x).**

   Example: Graph the function, $y = \frac{x^2 + 2}{1 - x}$ and since the numerator does not factor,

   a) there are no x-intercepts but there is a vertical odd asymptote at 1 and long dividing gives an oblique asymptote at y = -x - 1, so a quick graph is :
   b) We find f'(x) by using the quotient rule and simplifying

   $$\mathbf{f'(x) = \frac{(1-x)(2x)-(x^2+2)(-1)}{(1-x)^2} = \frac{2x - 2x^2 + x^2 + 2}{(1-x)^2} = \frac{-x^2 + 2x + 2}{(1-x)^2}}$$, so the critical values (real roots of the numerator) are: **x = 2.732, x = -.732** and from the denominator x = 1. Therefore the **critical points** are: **(-.732, 1.464) & (2.732, -5.464),** there is no point with x = 1 since it makes the denominator zero. By observation it is easy to see that the point (-.73, 1.46) is a relative minimum point, and (2.73, -5.46) is a relative maximum point. The next two steps confirm that conclusion.

c) We find f''(x) by taking the derivative of the first derivative using the quotient rule again:

$$\mathbf{f''(x)} = \frac{\mathbf{(1-x)^2(-2x+2) - (-x^2+2x+2)\cdot 2(1-x)(-1)}}{\mathbf{(1-x)^4}}$$, but the two terms on top have a common factor of $(1 - x)$ so f''(x) reduces to: $\frac{(1-x)(-2x+2) + 2(-x^2+2x+2)}{(1-x)^3}$ which simplifies as follows:

$$\frac{-2x + 2x^2 + 2 - 2x - 2x^2 + 4x + 4}{(1-x)^3} = \frac{6}{(1-x)^3}$$ and there are no inflection values from the numerator and 1 does not work since it makes the denominator zero. Therefore there are **no inflection points** (no point where the concavity changes.

d) 2nd derivative test confirms there will be a relative minimum point when x = -.73, since f''(-.73) is positive and that there will be a relative maximum point when x = 2.73, since f''(2.73) is negative.

2) Optimization games – any word problem that asks for a minimum or a maximum can be solved following these steps:

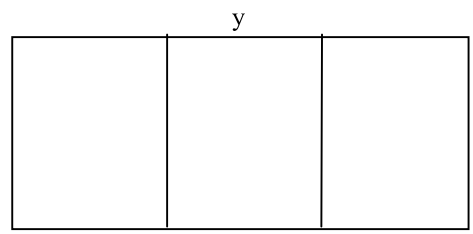
a) Identify all unknowns and **write** down **all the equations that might apply** to the given problem, then **combine** them into **one equation** having only **two variables** (one the optimization variable).

b) Find the **derivative**: $\frac{d(\text{opt})}{d(\text{other})}$ and find all **critical values**, plug in the original equation to find **critical points** if needed.

c) Find the **second derivative**: $\frac{d^2(\text{opt})}{d(\text{other})^2}$ and apply the **2nd derivative test** on the **critical values** from part b.

d) Answer the question that was asked.

Example: What is the minimum amount of fence needed to enclose 2000 ft^2 of pasture if the fence was placed as shown in the sketch?



a) Call the entire length of the 3 pastures 'y' and the width 'x'; the formulas that might apply here are length of fence: L = 2y + 4x and area: A = xy and since 2000 ft^2 is area: 2000 = xy so solving for y would give $y = \frac{2000}{x}$ and substituting into L gives: $L = 2\cdot\frac{2000}{x} + 4x$ which is one equation having only two variables and one of them is L the item to be minimized.

b) Find the derivative $\frac{dL}{dx}$ by first rewriting $L = 4000\,x^{-1} + 4x$ and taking the derivative:

$$\frac{dL}{dx} = -4000x^{-2} + 4 = \frac{-4000}{x^2} + 4 = \frac{-4000 + 4x^2}{x^2}$$, the **cv's** are the roots of the numerator **x = 31.62 and -31.62** and the root of the denominator x = 0 (which makes L undefined), the critical points are (31.62, 252.98) and (-31.62, -252.98).

c) Find the second derivative: $\frac{d^2L}{dx^2} = 8000x^{-3} = \frac{8000}{x^3}$, plugging in the cv 31.62 gives a positive and so the critical point (31.62, 252.98) will be a minimum point as desired. Note: the other cv, -31.62 gives a negative and would therefore give a maximum point but x cannot be negative anyway.

d) Now the question asked what the minimum amount of fence would be and that answer is the y value from the minimum point (can you see why?). Therefore the answer to the question is: The **minimum amount of fence** needed is **252.98 feet**.

C) **L'Hopital's Rule** – because of the derivative and consequences of the derivative like the **Mean Value Theorem** (you are to read about this theorem in your text), it is possible to prove another way to find limits if plugging in the 'a' gives the indeterminate forms: $\frac{0}{0}$ or $\frac{\infty}{\infty}$. **The Rule is**: $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Examples: $\lim_{h\to 0}\frac{\sin(h)}{h} = \lim_{h\to 0}\frac{\cos(h)}{1} = 1$ and $\lim_{h\to 0}\frac{\cos(h)-1}{h} = \lim_{h\to 0}\frac{-\sin(h)}{1} = 0$, Note: the h is a variable like x.

Example 3: $\lim_{x\to\infty}\frac{x}{\ln x} = \lim_{x\to\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty$, what is the limit if the fraction was flipped over?

Note: Section 4.6 of your text discusses other indeterminate forms: $0 \cdot \infty$, 0^0, 1^∞, and $\infty - \infty$; the first of these can always be manipulated into $\frac{0}{0}$ or $\frac{\infty}{\infty}$ but the last three are easier to do by graphing and preparing a table.