Math 112 Lecture IV: Integrals - Indefinite and Definite, another limit is reached.

A) Indefinite Integrals are really just Anti-derivatives or the derivative rules backwards but you need to know that $\int 1dx = x$, or $\int 1dp = p$. This new symbol, $\int \mathbf{1dx = x}$, is read "The **anti-derivative of 1** with **respect to x, is** equal to **x**" (of course the x's can be replaced with anything).

1) **Constant Rule:** $\int \mathbf{0dx = c,}$

   This rule comes from the constant derivative rule as follows: since $\frac{d(c)}{dx} = 0$, it follows that d(c) = 0 dx and taking the integral of both sides yields, $\int 1d(c) = \int 0dx$, but the left side equals c, therefore: $\int 0dx = c$. Note: the form d(c) = 0 dx is called the differential form of the derivative.

2) **Power Rule:** $\int \mathbf{x^n dx = \frac{x^{n+1}}{n+1}}$,

   Since $\frac{d(x^n)}{dx} = nx^{n-1}$, the differential is $d(x^n) = nx^{n-1}dx$ and taking the integral gives $\int 1d(x^n) = \int nx^{n-1}dx$. The left member is just x^n and since n is a constant inside the integral it can come out in front giving: $x^n = n\int x^{n-1}dx$. It is now possible to divide both sides by n getting: $\int x^{n-1}dx = \frac{x^n}{n}$, and if n was n + 1: $\int \mathbf{x^n dx = \frac{x^{n+1}}{n+1}}$. Note: constants in derivatives and in anti-derivatives multiply by the answers, so $\int \mathbf{5dx} = 5\int 1d\mathbf{x} = \mathbf{5}x$.

3) **Sum Rule:** $\int \mathbf{f + gdx = \int fdx + \int gdx}$

   Because of this rule every function can be changed by adding a '**+ 0**' inside the two pieces of bread, $\int$ *and* dx, and therefore, the anti-derivative will include a '**+ c**': $\int x^2dx = \int x^2 + 0dx = \int x^2dx + \int 0dx = \frac{x^3}{3} + c$. Because of this **every anti-derivative must include '+ c'**.

4) Product Rule: none now, later this will give 'Integration by parts', part D of this lecture.

5) Quotient Rule: none ever.

6) Trig rules there are six but they are backwards, so **the answers are the six trig functions**, and the negative ones are the cofunctions:
   a) $\int \mathbf{\cos x dx = \sin x + c}$
   b) $\int \mathbf{\sin x dx = -\cos x + c}$
   c) $\int \mathbf{\sec^2 x dx = \tan x + c}$
   d) $\int \mathbf{\csc^2 x dx = -\cot x + c}$
   e) $\int \mathbf{\sec x \tan x dx = \sec x + c}$
   f) $\int \mathbf{\csc x \cot x dx = -\csc x + c}$

   We will show how the rule is developed for cot, all the rest are similar: Since the differential is d(cot x) = $-\csc^2$x dx, integrating both sides gives $\int \mathbf{1}d(\cot x) = \int -\csc^2 x dx$, but the left member is just cot x and since -1 is a constant both sides can be multiplied by -1 and the rule is obtained: $\int \mathbf{\csc^2 x dx = -\cot x + c}$

7) Exponential and logs still have three:
   a) $\int \mathbf{e^x dx = e^x + c}$
   b) $\int \mathbf{a^x dx = \frac{a^x}{\ln a} + c}$
   c) $\int \mathbf{\frac{1}{x} dx = \ln x + c}$

   Since $\frac{d(a^x)}{dx} = a^x \ln a$, the integral of it's differential is $\int 1d(a^x) = \int a^x \ln a dx$. The left side is a^x and since 'ln a' is a constant: $a^x = \ln a \int a^x dx$, now dividing by 'ln a' gives: $\int \mathbf{a^x dx = \frac{a^x}{\ln a} + c}$

   Note: The last rule could be written: $\int x^{-1}dx$ and therefore the temptation is to use the power rule, why can't you use the power rule?

8) The Chain Rule backwards is called the 'Parable of the Sheep' by Bro. Saunders, it is called 'u substitution' in the text. The object of this rule is to identify a 'u', find the derivative and solve for 'du', substitute both 'u' and 'du' into the problem changing the variables to 'u', if you have selected the right 'u' you will now have one of your rules in terms of u.

Example1: $\int tan\, x dx = \int \frac{sin\, x}{cos\, x} dx$, let u = cos x, $\frac{du}{dx} = -sin\, x$ so $du = -sin\, x\, dx$, now if a negative is multiplied inside and outside the integral symbol: $\int tanxdx = -\int \frac{-sinx}{cosx} dx = -\int \frac{1}{u} du = -ln(u) + c$ or $= -ln(cosx) + c$, because of our log rule the negative in front of ln can be written as a power, so –ln(cos x) = ln(cos x)$^{-1}$ = ln(sec x). Therefore the integral of tan x with respect to x is ln(sec x) + c.

Example 2: $\int 2x^2\sqrt{4-2x^3}\,dx$ if u = 4 – 2x^3 (the function inside another function), du = -6x^2 dx and by multiplying and dividing by -3 and substituting:

$$\frac{-3}{-3}\int \mathbf{2x^2\sqrt{4-2x^3}dx} = \frac{1}{-3}\int -6x^2\sqrt{4-2x^3}\,dx = -\frac{1}{3}\int \sqrt{u}du = -\frac{1}{3}\frac{u^{3/2}}{3/2} = -\frac{2}{9}u^{\frac{3}{2}} + c = -\frac{2}{9}(4-2x^3)^{\frac{3}{2}} + c$$

9) The 6 Inverse Trig functions, their derivatives and their anti-derivatives:

a) $\left(sin^{-1} x\right)' = \frac{\mathbf{1}}{\sqrt{\mathbf{1}-x^2}}$

b) $\left(cos^{-1} x\right)' = \frac{\mathbf{-1}}{\sqrt{\mathbf{1}-x^2}}$

c) $\left(tan^{-1} x\right)' = \frac{\mathbf{1}}{\mathbf{1}+x^2}$

d) $\left(cot^{-1} x\right)' = \frac{\mathbf{-1}}{\mathbf{1}+x^2}$

e) $\left(sec^{-1} x\right)' = \frac{\mathbf{1}}{x\sqrt{x^2-\mathbf{1}}}$

f) $\left(csc^{-1} x\right)' = \frac{\mathbf{-1}}{x\sqrt{x^2-\mathbf{1}}}$

Since the derivatives of the cofunctions are just the negative of the functions there are only 3 anti-derivative rules:

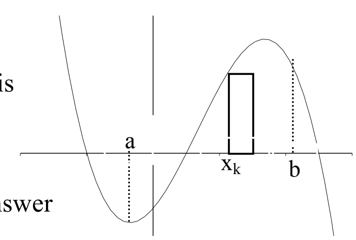
a) $\int \frac{1}{\sqrt{1-x^2}} dx = sin^{-1}x + c$

b) $\int \frac{1}{1+x^2} dx = tan^{-1}x + c$

c) $\int \frac{1}{x\sqrt{x^2-\mathbf{1}}} dx = sec^{-1}x + c$

B) The Definite Integral is the third item having two definitions: Definite Integrals are written $\int_a^b f(x)dx$, and the a and b are called the limits of the integral.

1) Definitions:



a) Intuitive definition of $\int_a^b f(x)dx$ is the **net area** between f(x) and the x-axis as x goes from a to b. If more area is above the x-axis the answer is **positive** (as in the graph to the right), if more is below the x-axis the answer is **negative**, and if the amount above and below is the same the answer is **zero**.

b) Math definition: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$, this definition can be understood by referring to the graph also.

The rectangle shown has a height of f(x_k) and a width of Δx (where $\Delta x = \frac{b-a}{n}$), imagine rectangles filling the area between x = a and x = b. The $\sum$ requires that we add the areas of all the rectangles, while the limit has us increase the number of rectangles to infinity. The resulting answer is the **exact area**, since n goes to infinity.

Example: Given the definite integral $\int_{-1}^{2} \mathbf{3}x - \mathbf{2}x^2 dx$ first we must determine Δx, $\Delta x = \frac{\mathbf{2}+\mathbf{1}}{n} = \frac{\mathbf{3}}{n}$, second we must determine x_k, x_0=a, x_1=a + Δx, so x_k = a + kΔx or in this case $x_k = -1 + k\frac{\mathbf{3}}{n}$ or $\mathbf{-1}+\frac{\mathbf{3}k}{n}$, that means f($x_x$) is

$f(-1+\frac{3k}{n}) = 3\left(-1+\frac{3k}{n}\right) - 2\left(-1+\frac{3k}{n}\right)^2 = -3+\frac{9k}{n} - 2\left(1-\frac{6k}{n}+\frac{9k^2}{n^2}\right) = -3+\frac{9k}{n}-2+\frac{12k}{n}-\frac{18k^2}{n^2}$ and simplifying f(x_k) = $-5+\frac{21k}{n}-\frac{18k^2}{n^2}$. Third to handle the summations, we need the following summation rules: $\sum_{k=1}^{n} c = n\cdot c$, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$, and $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$. So $\int_{-1}^{2} 3x-2x^2 dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x = \lim_{n\to\infty}\sum_{k=1}^{n}\left(-5+\frac{21k}{n}-\frac{18k^2}{n^2}\right)\frac{3}{n} = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{-15}{n}+\frac{63k}{n^2}-\frac{54k^2}{n^3}$ and using a summation rule for each term gives: $= \lim_{n\to\infty}\left(\frac{-15}{n}n+\frac{63}{n^2}\frac{n(n+1)}{2}-\frac{54}{n^3}\frac{n(n+1)(2n+1)}{6}\right)$ reducing and simplifying gives us: $= \lim_{n\to\infty}\left(-15+\frac{63}{2}+\frac{63}{2n}-18-\frac{27}{n}-\frac{9}{n^2}\right)$ and the limit is $-15+\frac{63}{2}-18 = -\frac{3}{2}$. So $\int_{-1}^{2} 3x-2x^2 dx = -\frac{3}{2}$, which is the net area between $3x - 2x^2$ and the x-axis as x goes from -1 to 2.

2) Other nice things to know:

a) The First Fundamental Theorem of Integral Calculus: If f(x) is continuous on the closed interval [a, b] (also necessary for the Math definition) and if F(x) is an anti-derivative of f(x) then $\int_a^b f(x)dx = F(b) - F(a)$.

Example: $\int_{-1}^{2} 3x-2x^2 dx$, an anti-derivative is F(x) = $\frac{3x^2}{2}-\frac{2x^3}{3}$. Therefore, by the First Fundamental Theorem of Integral Calculus $\int_{-1}^{2} 3x-2x^2 dx$ = F(2) – F(-1) = $\left(6-\frac{16}{3}\right)-\left(\frac{3}{2}+\frac{2}{3}\right) = 6-\frac{16}{3}-\frac{3}{2}-\frac{2}{3} = -\frac{3}{2}$, which is the same answer as we got above using the definition.

b) Switching the limits of integration changes the sign: $\int_a^b f(x)dx = -\int_b^a f(x)dx$. **Homework**, why is this true?

c) If f(x) is continuous over all intervals containing a, b, and c then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ and it does not matter whether c is between a and b or in front of both or behind both.

d) Total Area between f(x) and the x-axis as x goes from a to b: Total Area = $\int_a^b |f(x)|dx$

Example: What is the Total Area between the function, f(x) = $3x - 2x^2$ and the x-axis as x goes from -1 to 2? $\int_{-1}^{2} |3x-2x^2|dx$ = 3.75. Notice the total area must always be positive. **Homework**, why absolute value of f(x)?

e) Average Value of a function, f(x), as x goes from a to b: $f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$ Example: What is the average value of the function, f(x) = $3x - 2x^2$ as x goes from -1 to 2? $f_{ave} = \frac{1}{3}\int_{-1}^{2} 3x-2x^2 dx = \frac{1}{3}\left(-\frac{3}{2}\right) = -\frac{1}{2}$

f) The Second Fundamental Theorem of Integral Calculus: If f(x) is continuous on an open interval I containing a, then, for every x in the interval, $\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$

Example: $\frac{d}{dx}\left[\int_2^x 2t\sin t\,dt\right] = 2x\sin x$. Another example: $\frac{d}{dx}\left[\int_1^{x^2} \sin^2 t\,dt\right] = \sin^2 x^2 \cdot \frac{d(x^2)}{dx} = 2x\sin^2 x^2$

C) 12 New Functions and the calculus rules for them: 6 Hyperbolic Functions and 6 Inverse Hyperbolic Functions.

2) The 6 Hyperbolic functions, their derivatives and their anti-derivatives: It is first necessary to define them, we do this by defining $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ and by stating that all ratios and reciprocals are the same as for the trig functions. The rules now follow (prove them as **homework** using the definitions for sinh x & cosh x):

a) $(\sinh x)' = \cosh x$

b) $(\cosh x)' = \sinh x$

c) $(\tanh x)' = \text{sech}^2 x$

d) $(\coth x)' = -\text{csch}^2 x$

e) $(\text{sech}\, x)' = -\text{sech}\, x \tanh x$

f) $(\text{csch}\, x)' = -\text{csch}\, x \coth x$

The anti-derivative rules follow:

a) $\int \cosh x\, dx = \sinh x + c$

b) $\int \sinh x\, dx = \cosh x + c$

c) $\int \text{sech}^2 x\, dx = \tanh x + c$

d) $\int \text{csch}^2 x\, dx = -\coth x + c$

e) $\int \text{sech}\, x \tanh x\, dx = -\text{sech}\, x + c$

f) $\int \text{csch}\, x \coth x\, dx = -\text{csch}\, x + c$

3) The 6 Inverse Hyperbolic functions, the derivatives and their anti-derivatives: The proofs will require using the first identity for Hyperbolic functions: $\mathbf{\cosh^2 x - \sinh^2 x = 1,\ 1 - \tanh^2 x = sech^2 x,\ \coth^2 x - 1 = csch^2 x}$.

a) $\left(\sinh^{-1} x\right)' = \dfrac{1}{\sqrt{1+x^2}}$

b) $\left(\cosh^{-1} x\right)' = \dfrac{1}{\sqrt{x^2-1}}$

c) $\left(\tanh^{-1} x\right)' = \dfrac{1}{1-x^2}$ if $|x| < 1$

d) $\left(\coth^{-1} x\right)' = \dfrac{1}{1-x^2}$ if $|x| > 1$

e) $\left(\text{sech}^{-1} x\right)' = \dfrac{-1}{x\sqrt{1-x^2}}$

f) $\left(\text{csch}^{-1} x\right)' = \dfrac{-1}{|x|\sqrt{1+x^2}}$

The anti-derivative rules follow:

a) $\int \dfrac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1} x + c$

b) $\int \dfrac{1}{\sqrt{x^2-1}}\, dx = \cosh^{-1} x + c$

c) $\int \dfrac{1}{1-x^2}\, dx = \tanh^{-1} x + c$, if $|x| < 1$; $\coth^{-1} x$ if $|x| > 1$

d) $\int \dfrac{1}{x\sqrt{1-x^2}}\, dx = -\text{sech}^{-1} x + c$

e) $\int \dfrac{1}{x\sqrt{1+x^2}}\, dx = -\text{csch}^{-1}|x| + c$

Note: The proofs are similar to the following proof for $(\text{sech}^{-1}x)'$: $y = \text{sech}^{-1}x$ and taking the sech of both sides gives, sech y = x, now take the derivative with respect to x, $\mathbf{-\text{sech}\, y \tanh y \dfrac{dy}{dx} = 1}$, solving for dy/dx and substituting x for sech y and with tanh y = $\sqrt{1-\text{sech}^2 y}$ gives $\dfrac{dy}{dx} = \dfrac{-1}{x\sqrt{1-x^2}}$. Prove the rest as **homework**.

D) Integration By Parts or the Product Rule revisited. Suppose you have the product (u·v) where both u and v are functions of x. The product rule then gives, $\mathbf{\dfrac{d(uv)}{dx} = v\dfrac{du}{dx} + u\dfrac{dv}{dx}}$ multiplying by dx gives the differential form: d(uv) = v·du + u·dv and integrating both sides gives, $\mathbf{\int 1 d(uv) = \int v du + \int u dv}$. But left side is just **u·v** and if we solve for $\mathbf{\int u dv}$ we have the formula for integration by parts: $\mathbf{\int u dv = u \cdot v - \int v du}$. This rule allows us to take an anti-derivative we cannot integrate and change it to another anti-derivative that we can integrate, by selecting a 'u' and a 'dv' and finding 'du' and 'v'.

Example 1: $\mathbf{\int x \cdot \sin x\, dx}$, it can't be integrated by past methods but if u = x and dv = sin x dx, it is possible to get du = dx and v = -cos x. Therefore $\mathbf{\int x \cdot \sin x\, dx = -x \cdot \cos x - \int -\cos x\, dx}$ which equals $\mathbf{-x \cdot \cos x + \sin x + c}$.

Example 2: $\mathbf{\int x^4 \ln x\, dx}$, let u = ln x, and dv = x^4dx, therefore du = $\mathbf{\dfrac{1}{x}dx}$, and v = $\mathbf{\dfrac{x^5}{5}}$. Now using the formula gives us

$$\int x^4 \ln x\, dx = \frac{x^5}{5}\ln x - \int \frac{x^5}{5}\frac{1}{x}dx = \frac{x^5}{5}\ln x - \frac{1}{5}\int x^4 dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + c.$$