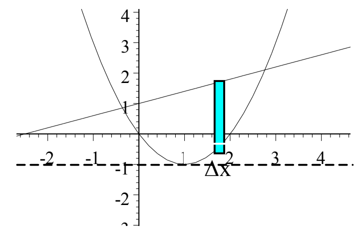
Math 112 – Lecture V: Applications of Integration both kinds

A) Applications of the definite integral all focus on the definition particularly $f(x_k)$ and Δx. Each application requires finding a sample piece and identifying $f(x_k)$ and Δx.

1) **Area between curves** assigns the same meaning to $f(x_k)$ and Δx as the definition does, **height is $f(x_k)$**, and **width is Δx** (illustrated in the graph to the right). In the graph the parabola is $f(x) = x^2 - 2x$ and the straight line is $g(x) = .4x + 1$. Therefore, the sample rectangle has a height $f(x_k) = g(x) - f(x)$, since $g(x)$ is on top and $f(x)$ is on the bottom of the rectangle, and the sample rectangle has a width of Δx. Now imagine rectangles filling up the area between the two curves, the sums of the areas of all rectangles would approximate the area between the curves but the integral $\int_a^b \mathbf{g(x) - f(x)dx}$ would give the exact area. The a and b are the x values where the two functions intersect, in this case where $x^2 - 2x = .4x + 1$ or where $0 = 1 + 2.4x - x^2$, therefore a = -.36 and b = 2.76 (using 'poly'). The exact area then is $\int_{-.36}^{2.76} \mathbf{1 + 2.4x - x^2 dx}$ = 5.08 square units of area.



2) **Volumes of rotation** are done **two** ways and each way assigns to **$f(x_k)$, area** and to **Δx, thickness**.

a) **Disk or Washer** Method – a region like that between the functions in part 1 is rotated about the line y = -1, generating a three dimensional solid similar to diagrams on page 424 of the text. To find the volume generated:

i) **Select a sample piece** (a cross section with width Δx) **perpendicula**r to the axis y = -1. Each sample piece will be a circle with a hole in it for every place except when x is '1' and it's width will be Δx.

ii) Find the **volume of the sample piece**: $\mathbf{\Delta V = (\pi R^2 - \pi r^2)\,\Delta x}$, where R is the radius (or distance) from the axis (-1) to the top function, $.4x + 1$, and r is the radius from the axis (-1) to the bottom function, $x^2 - 2x$. So the volume is $\Delta V = [\pi(.4x + 1 - (-1))^2 - \pi(x^2 - 2x - (-1))^2]\,\Delta x$.
Note: **vertical distances** are always '**y top – y bottom**', **horizontal distances** are always '**x right – x left**'.

iii) **Translate** from the sample piece **to a definite integral**: $\int_{-.36}^{2.76} \mathbf{\pi(.4x + 2)^2 - \pi(x^2 - 2x + 1)^2 dx}$, in the integral the Δx changes to dx and for this example the limits stay the same as for area between curves since sample pieces can be taken only as x goes from -.36 to 2.76.

iv) **Solve the integral**: $\int_{-.36}^{2.76} \mathbf{\pi(.4x + 2)^2 - \pi(x^2 - 2x + 1)^2 dx}$ = 48.023 cubic units of volume.

b) **Tin Can or Shell** Method – a region like that between the functions in part 1 is rotated about the line x = -1, generating a three dimensional solid. To find the volume generated:

i) **Select a sample rectangle** (like the colored rectangle in part 1 above) **before rotation**, it must be **parallel** to the axis of rotation, x = -1. This rectangle when rotated about the axis forms a tin can without a top or bottom, the thickness of the tin is Δx and the height of the can is $g(x) - f(x)$.

j) **Find the volume of tin in the sample can**: ΔV = height · circumference · thickness = $[g(x) - f(x)]\, 2\pi r\, \Delta x$, where 'r' is the horizontal distance from the sample rectangle to the axis or '**x right – x left**' and since the sample rectangle's x value is always changing we let 'x right' = x, but the axis of rotation does not change it is always x = -1, therefore 'r' = x – (-1) = x + 1 and $\Delta V = [(.4x + 1) - (x^2 - 2x)]\, 2\pi(x + 1)\, \Delta x$ and simplifying: $\Delta V = 2\pi\,(1 + 2.4x - x^2)\,(x+1)\,\Delta x$

k) **Translate into a definite integral**. Imagine infinitely many tin cans filling up the solid and you get the definite integral: $\int_{-.36}^{2.76} \mathbf{2\pi(1 + 2.4x - x^2)(x + 1)dx}$

l) **Solve the integral**: $\int_{-.36}^{2.76} \mathbf{2\pi(1 + 2.4x - x^2)(x + 1)dx}$ = 35.123 cubic units for volume. Note: this answer is not the same as the disk method answer because for that problem the region was rotated about the line y = -1 and for this problem the region was rotated about the line x = -1 getting a different solid.

Note: For the examples that were chosen for each method it would be **impossible** to solve them using the other method because the sample piece would have had **the same function on both ends** of the rectangle. The volume of some regions can be found using either the Disk or the Tin Can method but only when the sample piece touches two separate curves at both ends.

3) Arc Length and Surface Area, for each of these the Δx is really Δs, where $\mathbf{\Delta s \approx \sqrt{\left(\frac{\Delta y}{\Delta x}\right)^2 + 1}\,\Delta x}$ or $\mathbf{\Delta s \approx \sqrt{\left(\frac{\Delta x}{\Delta y}\right)^2 + 1}\,\Delta y}$. **In the integral** the Δ**s** changes into **ds** and $\mathbf{\frac{\Delta y}{\Delta x}}$ changes to $\mathbf{\frac{dy}{dx}}$ which is the derivative with respect to x (the flipped over version is merely the derivative with respect to y).

a) **Arc Length** assigns **to f(x_k)** the value **'1'** and assigns **to Δx** the **'Δs'**. Therefore, to find the length of arc for the parabola in part 1, y = x^2 – 2x, as x **goes** from 0 to 3, you would:

i) **Select** a sample piece of **arc**, Δ**s** which turns into **ds** in the integral and equals: $\mathbf{ds} = \sqrt{\left(\frac{\mathbf{dy}}{\mathbf{dx}}\right)^2 + \mathbf{1}}\,\mathbf{dx}$.

j) **Find** $\frac{\mathbf{dy}}{\mathbf{dx}}$ and **square** it: $\frac{\mathbf{dy}}{\mathbf{dx}}$ = 2x – 2, squaring it gives $4x^2 - 8x + 4$, so $\mathbf{ds} = \sqrt{\mathbf{4x^2 - 8x + 4 + 1}}\,\mathbf{dx}$.

k) **Translate** into a **definite integral**: $\int_0^3 \sqrt{\mathbf{4x^2 - 8x + 5}}\,\mathbf{dx}$, notice the limits are 0 and 3 since it is in the 'dx' form and x goes from 0 to 3.

l) **Solve** the integral: $\int_0^3 \sqrt{\mathbf{4x^2 - 8x + 5}}\,\mathbf{dx}$ = 6.126 units of length.

b) **Surface Area** assigns to **f(x_k)** the **circumference of the rotated arc**, 2πr, and to Δ**x** the **'Δs'**. Therefore, if the arc in the problem above was rotated about the line x = 3 getting a sombrero like figure, you would:

i) **Select** a sample piece of arc, Δ**s**, and **rotate it** about the line x = 3.

j) **Find the area** of the sample piece rotated about the axis: **Use** the **ds** found for arc length **and find 'r'**, the distance from the piece Δ**s** to the **axis of rotation x = 3.** This distance 'r' is a horizontal distance for this problem and therefore, it is 'x right – x left' or in this case r = 3 – x, since the axis of rotation is to the right of the piece of arc. Therefore ΔA $\approx$ **2πr ds = 2π (3 – x)** $\sqrt{\mathbf{4x^2 - 8x + 4 + 1}}\,\mathbf{dx}$.

k) **Translate** into a **definite integral:** $\int_0^3 \mathbf{2\pi(3 - x)}\sqrt{\mathbf{4x^2 - 8x + 5}}\,\mathbf{dx}$ and **solve** getting**: 46.132** square units.

Note: The dx form was used for both Arc Length and Surface Area since the function y = x^2 – 2x did not repeat x-values as x goes from 0 to 3, but it did repeat y-values so the dy form could not be used.

4) Work problems can be done by considering the summation of increments of work in the form Δ**W = F Δx,** or force times increments of distance (used when springs are compressed or stretched) but there is another way to formulate the increments of work in the form Δ**W = (ΔF) x,** or increments of force times distance (used in problems involving non-rigid substances like fluids and chains).

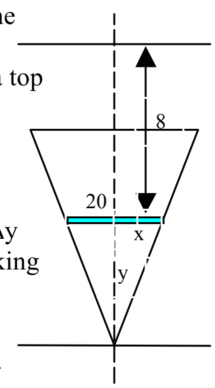
a) Type 1: Δ**W = F Δx**, it is important here to find F(x) by using Hooke's Law F(x) = kx.

Suppose the natural length of a spring is 9 inches and suppose that a 4 pound weight stretches it to 12 inches; how much work is done to stretch the spring to 16 inches?

First we find F(x) by observing that 4 pounds stretched the spring 3 inches beyond it's natural length, so from Hooke's Law F(3) = k·3 = 4, giving k = 4/3 and therefore F(x) = $\frac{\mathbf{4}}{\mathbf{3}}\mathbf{x}$.

Now since Δ**W = F Δx** it follows that W = $\int_0^7 \mathbf{F(x)dx} = \int_0^7 \frac{\mathbf{4}}{\mathbf{3}}\mathbf{xdx} = \mathbf{16.667}$ inch-pounds. Notice the limits on the definite integral are 0 to 7, because the question asked how much work was done to stretch the spring to 16 inches, just 7 inches beyond it's natural length.

b) Type 2: Δ**W = (ΔF) x**, it is important here to realize that ΔF is really ΔV times the density of the liquid (if water, it is 62.4 lbs per cubic foot); the x is the distance the ΔF is moved.



Suppose you have a water tank in the shape of a right circular cone that is 20 feet tall with a top radius of 8 feet, further suppose that it is full of water, how much work is required to pump the water out of the tank to a height 6 feet above the tank?

First we select a piece of water (the shaded piece in the figure) and determine ΔF for that piece of water, ΔF = ΔV·62.4 = 62.4 (π x^2) Δy, where Δy is the thickness of the the piece of water and x is the radius of the piece of water. Since the variables do not agree and since the Δy determines the eventual integral we must change the x into a formula with y, we do this by looking at the similar triangles $\frac{\mathbf{y}}{\mathbf{x}} = \frac{\mathbf{20}}{\mathbf{8}}$, so x = $\frac{\mathbf{2}}{\mathbf{5}}\mathbf{y}$, therefore ΔF = 62.4π $\frac{\mathbf{4}}{\mathbf{25}}\mathbf{y^2}$ Δy.

Now since Δ**W = (ΔF) x**, and x is the distance the piece of water is moved, in this case 26 – y (can you see why?), it follows that W = $\int_0^{20} \mathbf{62.4\pi \frac{4}{25} y^2 dy(26 - y)}$ which must be written $\int_0^{20} \mathbf{62.4\pi \frac{4}{25} y^2 (26 - y)\,dy}$ to solve and therefore the work required to lift all the water in the tank to a height 6 feet above the tank is 920059.39 foot-pounds. Notice the limits on the integral were 0 to 20 since there was water from 0 feet to 20 feet, if the tank had only been half full the limits would have been 0 to 10.

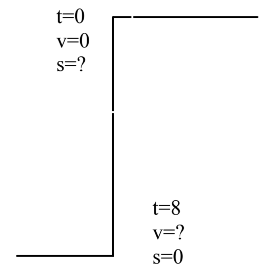
B) Applications of the indefinite integral lead to finding meaning for the '+ c'.

1) Deriving the formula, $A = Pe^{kt}$. This application comes from the fact that in some cases the derivative of a function is equal to a constant times the original function or in formula form $\frac{dy}{dx} = ky$.

Changing to differential form gives: dy = ky dx, which has more meaning by **separating the variables** by dividing by y, $\frac{dy}{y} = kdx$. Integrating both sides yields: $\ln|y| = kx + c$. Using the definition of logs gives us $|y| = e^{kx+c}$, but it is easily seen that the right side can never be negative so $y = e^{kx+c}$. It is possible to write the right side as $e^{kx}e^{c}$ because of the rules for exponents and e^c is just a constant we can call C. Therefore $y = Ce^{kx}$. It should be apparent that this formula is very similar to the formula $A = Pe^{kt}$ if we rename a few letters.

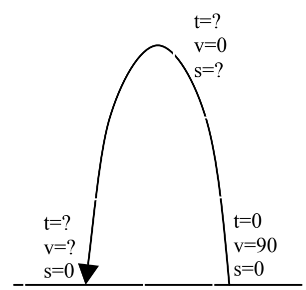
2) Acceleration (a), Velocity (v), and Position (s) are related as derivatives of each other respectively: $\frac{ds}{dt} = v$; $\frac{dv}{dt} = a$. Therefore they are also related as anti-derivatives: $v(t) = a \cdot t + c_1$; $s(t) = a\frac{t^2}{2} + c_1 t + c_2$. The constants come because of the indefinite integral.

Example 1: Suppose you were at the top of a cliff and you dropped a rock and it took 8 seconds to hit the ground below the cliff, how high would the cliff be?



Acceleration due to gravity is -32 feet per second squared or -9.8 meters per second squared. We will use the feet form for our example. Looking at the formulas for v(t) and s(t) above we substitute -32 for a and get: $v(t) = -32t + c_1$; $s(t) = -32\frac{t^2}{2} + c_1 t + c_2$ or $s(t) = -16t^2 + c_1 t + c_2$. At the beginning identify what t, v, and s are and do the same at the end, if they are not known a ? is placed there. Now since v = 0 when time is zero c_1=0 and therefore $v(t) = -32t$ and $s(t) = -16t^2 + c_2$, furthermore, when the rock hits the ground the position is zero and since it took 8 seconds we have $0 = -16(8^2) + c_2$ and therefore, $c_2 = 1024$ and so $s(t) = -16t^2 + 1024$ and so when t = 0 , s = 1024 and the cliff is 1024 feet high.

Example 2: Suppose you threw a rock into the air at 90 feet per second, how high will it go and when will it hit the ground?



In this problem we have three moments of interest, when the rock is tossed, when it reaches it's highest point and when it hits the ground, so we identify t, v, and s at all three moments. Since all three are known at time equals zero both constants can be found: $90 = -32(0) + c_1$, so $c_1 = 90$; and $0 = -16(0) + 90(0) + c_2$, so $c_2 = 0$. Therefore, the formulas are: $v(t) = -32t + 90$ and $s(t) = -16t^2 + 90t$. Now to answer the questions. It reaches it's highest point when v = 0, so 0 = -32t + 90 or t = 90/32 = 2.81 sec and plugging this time into s(t) gives $s = -16(2.81)^2 + 90(2.81) =$ **126.56 feet**. And to find out how long it takes before it hits the ground we set the s formula to zero and factor or use poly getting: $0 = -16t^2 + 90t$, so t = 0 and t = **5.625 seconds**.

3) Business applications are of interest too.

a) Connections – In business there are three related concepts: **Revenue, Cost** and **Profit**.

i) **Revenue** is the total amount of money that comes to a business as they sell products. When the revenue obtained from the sell of products is dependent on the number of products sold it is the 'price times the number sold'. In formula form **R(x) = p(x)·x**, where p(x) is the price function and x is the number of products sold. Example: If the price function was p(x) = 30 – .05x, then **R(x) = (30 – .05x)·x**

j) **Cost** is the amount of money needed to produce and sell products. **C(x) = 500 + 20x** could represent a monthly rent of $500 plus the cost of $20 to produce each product to sell.

k) **Profit** is Revenue – Cost or **P(x) = R(x) – C(x)** or for the examples given, $P(x) = 30x - .05x^2 - 500 - 20x$ or simplified, $\mathbf{P(x) = 10x - .05x^2 - 500}$.

b) **Derivatives** – The derivatives of the three concepts are called **Marginal** Revenue, **Marginal** Cost, and **Marginal** Profit respectively and are used to approximate the revenue, cost, or profit from the 20th item. For example if you were told that R'(20) = 150, you would know that the approximate revenue received in selling the 20th item would be $150.

c) **Anti-derivatives** – From the Marginal Revenue or Marginal Cost it is possible to get the Revenue and/or Cost function if you have other data too:

Example: If the price function was p(x) = 20 – .08x, and marginal cost was 12, what is the revenue function, the cost function if the cost was $650 for 20 items, and the profit function?

Since Revenue equals 'price times number of products', $R(x) = (20 - .08x)x = 20x - .08x^2$. Also since Marginal Cost is 12, $\mathbf{\frac{dC}{dx} = 12}$ which in differential form is, $dC = 12\,dx$ and integrating both sides gives $C(x) = 12x + C$, where this C is not cost, but we are given that $C(20) = 650 = 12(20) + C$, so $C = 410$ and therefore, $C(x) = 12x + 410$. Now since we know that Profit = Revenue – Cost, $P(x) = 8x - .08x^2 - 410$.

Furthermore we could now ask how many products would return a Maximum Revenue or a Maximum Profit? To get Maximums we must find the Marginal (derivative) Revenue and the Marginal Profit:
$R'(x) = 20 - .16x$ and the critical value is 125, so selling 125 items will yield a maximum Revenue;
$P'(x) = 8 - .16x$ and the critical vulue is 50, so selling 50 items will yield a maximum Profit. Why is this true?

C) Definite Integrals previously were classified as undefined if f(x) was not continuous on the closed interval [a, b] but we are now ready to consider improper integrals. They come in two types:

Type 1: a or b are negative or positive infinity respectively or both

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{b} f(x)\,dx \quad \text{or} \quad \int_{a}^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_{a}^{b} f(x)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty}\int_{a}^{c} f(x)\,dx + \lim_{b\to\infty}\int_{c}^{b} f(x)\,dx$$

$$\text{example: } \int_{-\infty}^{1}\frac{1}{x^2+1}\,dx = \lim_{a\to-\infty}\int_{a}^{1}\frac{1}{x^2+1}\,dx = \lim_{a\to-\infty}\tan^{-1}x\Big]_{a}^{1} = \lim_{a\to-\infty}(\tan^{-1}1 - \tan^{-1}a) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$$

Type 2: the function becomes infinite

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to a^+}\int_{c}^{b} f(x)\,dx, \text{ f(x) discontinuous at a;} \quad \text{or} \quad \int_{a}^{b} f(x)\,dx = \lim_{c\to b^-}\int_{a}^{c} f(x)\,dx, \text{ f(x) discontinuous at b;} \quad \text{or}$$

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to d^+}\int_{a}^{c} f(x)\,dx + \lim_{c\to d^-}\int_{c}^{b} f(x)\,dx, \text{ f(x) discontinuous at some point d, between a and b.}$$

Both types are illustrated on page 614 of the text.