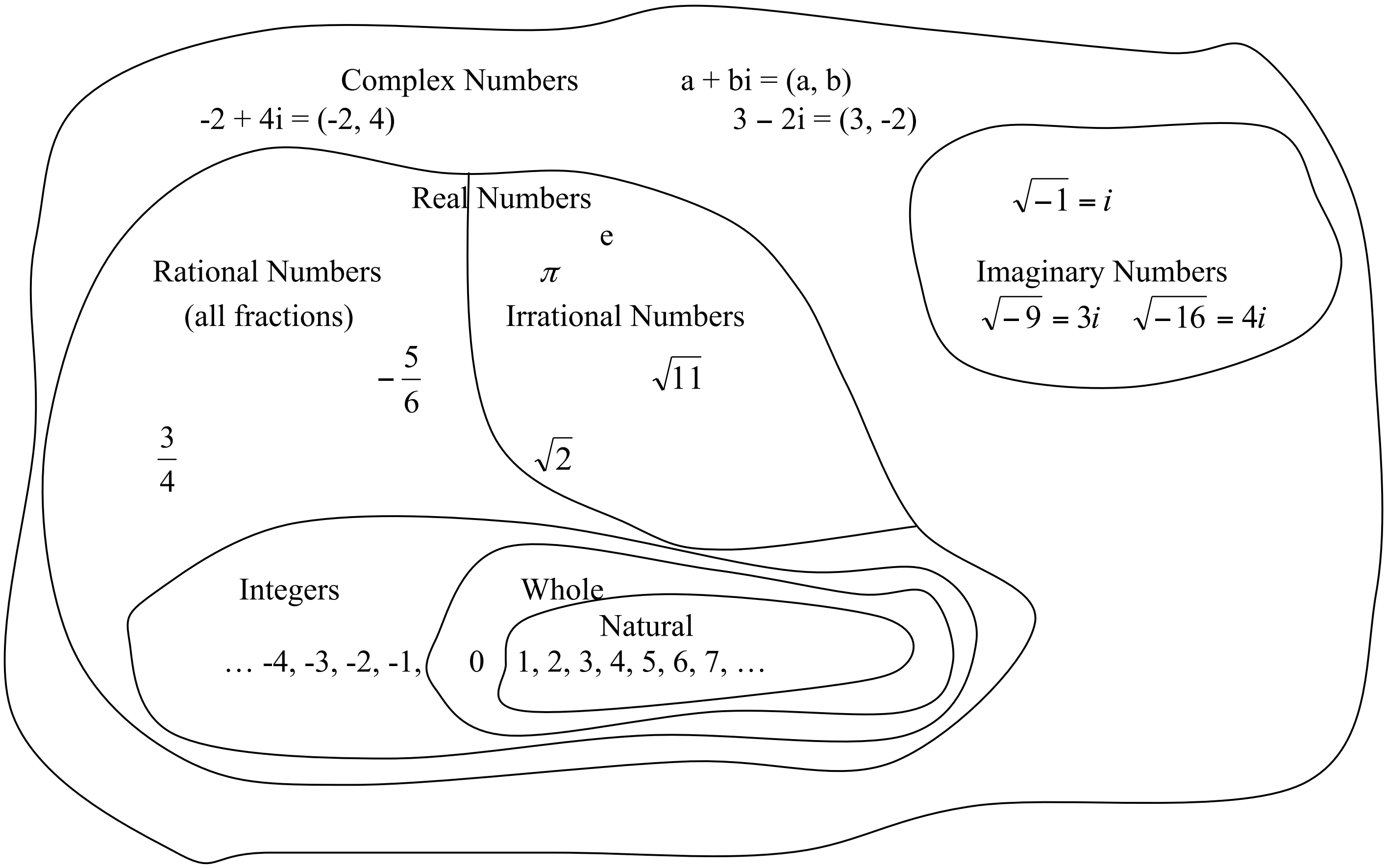
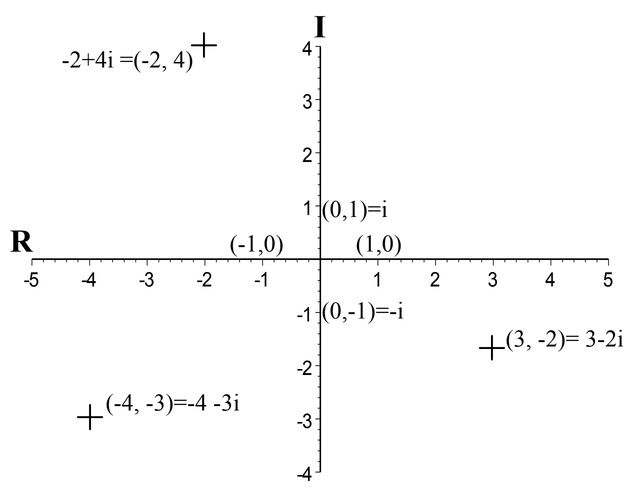
Lecture I – Review of Algebra, Plus

A) Complex Numbers – the numbers we use in Algebra

1) Venn Diagram



2) Complex Plane – Horizontal Axis has **Real** Numbers, Vertical Axis has **Imaginary** Numbers



3) The Operations of Addition, Subtraction, Multiplication, Division and Powers with Complex Numbers:

a) Addition and Subtraction are very similar and can both be done **by hand** if the Complex Numbers are in term form; i.e. (3 – 4i) **+** (-2 + 5i) **–** (1 – i) = 3 – 4i – 2 + 5i – 1 + i = 2i **By Calculator** they must be in ordered pair form for the TI 85 and 86, but in term form for the TI 83 and 89.

b) Multiplication **by hand** again requires them to be in term form and you simply foil them as you would binomials; i.e. (3 – 4i) (-2 + 5i) = -6 + 15i + 8i – 20i^2 but since $\mathbf{i^2 = -1}$ your answer simplifies to: -6 + 15i + 8i – 20(-1) or –6 + 15i + 8i + 20 = 14 + 23i after adding like terms. **By Calculator** they must be in ordered pair form for the TI 85 and 86, term form for the TI 83 or 89.

c) Division **by hand** requires the term form as well, plus **Multiplication by ONE**; i.e.

$$\frac{3-4i}{-2+5i}\cdot 1=\frac{3-4i}{-2+5i}\cdot\frac{-2-5\mathrm{i}}{-2-5\mathrm{i}}=\frac{-6-15i+8i+20i^2}{4+10i-10i-25i^2}=\frac{-26-7i}{29}=\frac{-26}{29}-\frac{7}{29}i$$

Notice the name for **ONE** must be the **conjugate of the denominator over itself**. **By Calculator** they must be in ordered pair form for the TI 85 and 86, term form for the TI 83 and 89.

d) Powers of i are found easily by referring to the four points on the axis of the complex plane that are one unit from the origin (0, 0). The four points are (1, 0), (0, 1), (-1, 0) and (0, -1) or in term form they are **1, i, -1, -i**. Therefore i to any power must be one of these; i.e. $i^0=1$, $i^1=i$, $i^2=-1$ and $i^3=-i$. The trick is to take the given power, divide it by 4, look at the remainder and give the answer. i^{23} then equals –i, $i^{142}=-1$, can you see why?

4) Absolute Value or Magnitude of a number – two types:

a) The **Absolute Value of a Complex Number** is the **distance** that number is from (0,0) on the Complex Plane:

$$|3-4i|=|(3,-4)|=\sqrt{3^2+(-4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

or $|-2+5i|=|(-2,5)|=\sqrt{(-2)^2+5^2}=\sqrt{4+25}=\sqrt{29}$

b) The **Absolute Value of a Real Number**, x (or any expression with x, like 3-2x): $|x|=\begin{cases}\text{x if x is positive}\\ \text{- x if x is negative}\end{cases}$

$$|3-2x|=\begin{cases}\text{3 - 2x if } 3-2x \text{ is positive}\\ \text{- 3 + 2x if 3 - 2x is negative}\end{cases}$$ (we often ignore the part after 'if')

This definition is used to write **two expressions** (the expression and it's negative) every time a problem has an absolute value with x; i.e. **|4x-5|** must be replaced with **(4x-5)** and **(-4x+5)**.

Notice: Whenever an absolute value has an **i** in it, your answer is **a distance** (a number); whenever an absolute value has an **x** in it, your answer has **two** expressions.

B) Functions – expressions with **two** variables (generally x and y with y = f(x)). There is a further restriction that for every value for x, one and only one y value can be obtained (**vertical line test**). The set of all numbers that can be substituted for x is called the **Domain**. The set of all number values y can have is called the **Range**. Three types of functions with an example are: <u>Linear</u>, $y = 3x + 2$; <u>Quadratic</u>, $y = 2x^2 - 3x - 4$; <u>Absolute Value</u>, $y = |2 + x|-3$. Note: y could be replaced with f(x) or g(x) or h(x).

1) **Equations** result when the y is replaced with any real number; **Inequalities** result when the y is replaced with any real number and the "equals" sign is replaced with one of the following: <, >, ≤, or ≥.

a) Equations - for each function above let y=2 and solve:

Linear: $2 = 3x + 2$ subtract 2 from both sides, $0 = 3x$ divide by 3, $0 = x$ so $x = 0$.

Quadratic: $2 = 2x^2 - 3x - 4$ subtract 2 from both sides, $0 = 2x^2 - 3x - 6$ and use the quadratic formula

$$x=\frac{3\pm\sqrt{9-4\cdot 2\cdot(-6)}}{4}=\frac{3\pm\sqrt{9+48}}{4}=\frac{3+\sqrt{57}}{4}\text{ or }\frac{3-\sqrt{57}}{4}$$ which gives x = 2.64 and x = –1.14.

Caution: If ever the number under the radical is negative the two solutions are complex numbers. If you have a TI 85 or 86 "poly" will give your answers immediately (find a program for the 83, "zeros" for 89)

Absolute Value: $2 = |2 + x| - 3$ add 3 to both sides, $5 = |2 + x|$ replace the absolute value with the two things it equals: $5 = 2 + x$ and $5 = -2 - x$ solve each equation getting, $3 = x$ and $7 = -x$ or $x = -7$ and check.

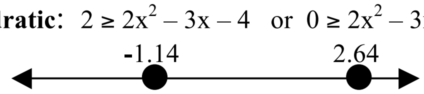
Caution: If ever an absolute value expression equals **a negative number** there are no solutions, why?

b) Inequalities - for each function above let y=2 and replace the "=" with "≥" or "<" and solve. All answers will be given in **Interval Notation**.

Linear: $2 \ge 3x + 2$ just like for equations you get, $0 \ge x$ or $x \le 0$ or in Interval Notation (-∞, 0].

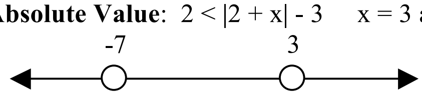
Caution: if ever you **multiply or divide by a negative number the sense must change** too.

Quadratic: $2 \ge 2x^2 - 3x - 4$ or $0 \ge 2x^2 - 3x - 6$ 2.64 and -1.14 are solutions giving dots on line,



and since substituting 0 for x gives $0 \ge -6$, a true statement, the solution is all values between and including the two points, or [-1.14, 2.64]

Absolute Value: $2 < |2 + x| - 3$ x = 3 and –7 were the equals solutions but now they are only circles



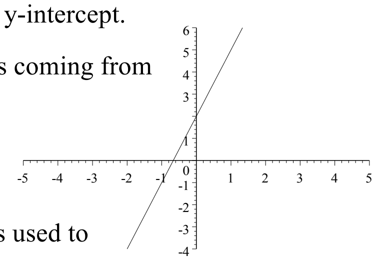
and since substituting 0 gives 2 < -1, a false statement, all values before –7 and after 3 are solutions, or in Interval Notation: (-∞, -7) or (3, ∞). The 'or' means 'union'. Notice a **'[' means 'include'** that point while A **'(' means 'do not include'**. Also note that '0' was chosen because it was between both points.

2) Graphing the three function types:

a) **Linear Functions** always have each variable raised only to the first power with the variables in different terms. Their graphs are always **straight lines**. Three forms are often encountered:

-**Intercept or Standard Form**: ax + by = c. Example: 2x – 3y = 6. The advantage of this form is the ease with which the x and y intercepts can be obtained: Just **let x = 0 to find the y-intercept**, for our example y = -2; and **let y = 0 to find the x-intercept**, for the example x = 3. So the graph is the straight line that crosses the y-axis at –2 and the x-axis at 3.

-**Slope-Intercept Form**: y = mx + b, where 'm' is the slope of the line and 'b' is the y-intercept.



The slope is defined as follows: $\text{slope} = m = \dfrac{\text{rise}}{\text{run}} = \dfrac{y_2 - y_1}{x_2 - x_1}$ with the x's and y's coming from any two points. Example: y = 3x + 2, the slope is 3 or more properly $\dfrac{3}{1}$ and the y-intercept is y = 2.

-**Point-Slope Form**: $y - y_1 = m(x - x_1)$, m is slope and (x_1, y_1) is one point. This is used to obtain a linear function when two points are given. Example: Find the linear function containing the points (-1, 5) and (2, 3). Procedure: Find m, $m = \dfrac{3-5}{2-(-1)} = \dfrac{-2}{3}$, and plug into the point-slope formula

along with one point, (2,3), giving: $y - 3 = \dfrac{-2}{3}(x-2)$ *and simplify* $y = \dfrac{-2}{3}x + \dfrac{4}{3} + 3\cdot\dfrac{3}{3}$ *or* $y = \dfrac{-2}{3}x + \dfrac{13}{3}$

b) **Quadratic Functions** always have a term with $\mathbf{x^2}$ but the y is not squared. Their graphs always give a **parabola**. To graph any quadratic function like $y = 2x^2 – 3x – 4$, follow the following **three steps**:

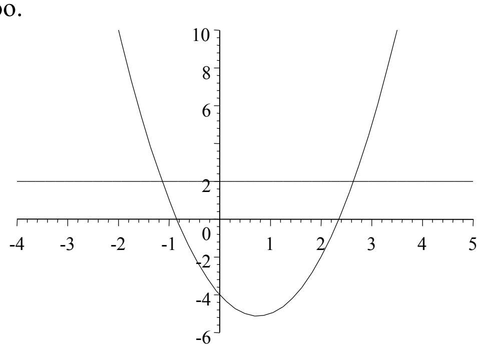
-Find the **y-intercept by letting x = 0**. In our example y = -4

-Find the **x-intercepts (if they exist) by letting y = 0** (this requires using the quadratic formula):

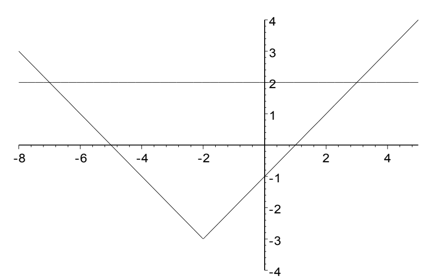
$$x = \frac{3 \pm \sqrt{9+32}}{4} = \frac{3}{4} \pm \frac{\sqrt{41}}{4} = 2.35 \text{ and } -.85$$ Note: the two solutions can be gotten from 'poly' too.

-Find the vertex from the quadratic formula results: $\left(\dfrac{3}{4}, \dfrac{-41}{8}\right)$ or (.75, 5.13).

In formula: $\left(\dfrac{-b}{2a}, \dfrac{-(b^2 - 4ac)}{4a}\right)$ Notice how parts of the quadratic formula are used, notice the differences too.



Notice the extra line at y = 2, the intersection points with that line and the parabola are the solutions obtained earlier. Also notice the y-intercept of the parabola is –4, the x-intercepts are -.85 and 2.35 and the vertex is (.75, 5.13). If the quadratic formula had given a negative under the radical, there would not have been any x-intercepts and the parabola would have been either totally above the x-axis or totally below it.



c) **Absolute Value Functions** always **bounce** and therefore they are like Quadratic functions and are graphed with the same **three** steps but the last two steps are reversed. Example y = |2 + x| - 3

-Find the **y-intercept** by letting x = 0. In our example **y = -1**

-Find the **vertex** (where the graph bounces) by letting the absolute value part, 2+x equal zero, solving for x, x = -2. Y then would be the remaining term or in this case –3. Therefore **the vertex is (-2, -3).**

-Find the **x-intercepts** (if they exist) by letting y = 0: 0 = |2 + x| - 3, adding 3 to both sides gives 3 = |2 + x|, now replacing the absolute value with it's two replacements gives 3 = 2 + x, and 3 = -2 – x, solving each of these equations gives **x = 1, and x = -5.** Since both do check in the original function giving y = 0 they are the x-intercepts.

3) Composition of functions and Inverses of functions.

a) **Composition** of functions is nothing more than the function of a function [for compositions y is always replaced with f(x) or g(x) or h(x)]. Example, suppose $f(x) = 3x - 2x^2$ and $g(x) = 4 - 3x$, then **f composite g** is:

$f \circ g = f(g(x)) = f(4 - 3x) = 3(4 - 3x) - 2(4 - 3x)^2 = 12 - 9x - 2(16 - 24x + 9x^2) = 12 - 9x - 32 + 48x - 18x^2$ $= -20 + 39x - 18x^2$ and **g composite f** is: $g \circ f = g(f(x)) = g(3x - 2x^2) = 4 - 3(3x - 2x^2) = 4 - 9x + 6x^2$.

b) **Inverses** exist only for functions that are **one-to-one**, which means that not only is there one and only one y for each x but there must also be one and only one x for each y (functions must pass a horizontal line test as well as the vertical line test). Obviously then both **quadratic** and **absolute value** functions are **not one-to-one** and therefore **do not have inverses**. All linear functions that are not horizontal are one-to-one and therefore **do have inverses**. Another function that does have an inverse is $h(x) = \dfrac{3x+1}{2-x}$, at this point you will have to trust me that h(x) is one-to-one since it is a linear function divided by another linear function.

You **find the inverse of a function, h(x), by**:

-Switching x and y and

-Solving for y

The result you get for **y is the inverse of h(x), written $h^{-1}(x)$.**

Example: find $h^{-1}(x)$. Switch x and y getting $x = \dfrac{3y+1}{2-y}$ and solve for y, $x(2-y) = \left(\dfrac{3y+1}{2-y}\right)(2-y)$, simplifying gives, $2x - xy = 3y + 1$, then adding xy and subtracting 1 gives, $2x - 1 = 3y + xy$ now by factoring out y on the right gives, $2x - 1 = (3 + x)\,y$ and dividing by (3 + x) gives, $y = \dfrac{2x-1}{3+x}$.

Therefore $\mathbf{h^{-1}(x)} = \dfrac{2x-1}{3+x}$.

c) Inverse Function Rule combines composition of functions with Inverses: $f \circ f^{-1} = x = f^{-1} \circ f$

Example using h(x) above: $h \circ h^{-1} = \dfrac{3\dfrac{2x-1}{3+x}+1}{2-\dfrac{2x-1}{3+x}} \cdot \dfrac{3+x}{3+x} = \dfrac{3(2x-1)+1(3+x)}{2(3+x)-(2x-1)} = \dfrac{6x-3+3+x}{6+2x-2x+1} = \dfrac{7x}{7} = x$.

You would also get x if you worked out $h^{-1} \circ h$, do this as a homework problem.

4) Solving inequalities using functions and graphs requires the following steps:

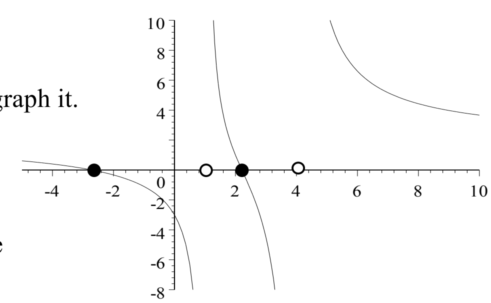
a) Set one side of the inequality to zero, call the other side y and graph it with your calculator.

b) Set each denominator expression to zero, solve for x and place a circle on the x-axis for each.

c) Find all x-intercepts, put a **dot** for each x-intercept **if equals** is in the inequality, put a **circle if not**.

d) Now look at the original inequality set to zero, is y > 0 or is y < 0; if **y > 0** think "for which x values is the graph **above the x-axis?**", identify all pieces and write your solution in Interval Notation.

Example: $\dfrac{2x}{x-4} \geq \dfrac{3}{1-x}$



a) Set the right side to zero $\dfrac{2x}{x-4} - \dfrac{3}{1-x} \geq 0$, call left side y, graph it.

b) x – 4 = 0 gives x = 4 and 1 – x = 0 gives x = 1, **circle** them.

c) Use the 'root' function on your grapher to find the x-int's
For this graph they are at –2.71 and 2.21, put **dots** there since 'equals' is included in the inequality.

d) From part b, y ≥ 0, so our solution is for all x where the graph is **above the x-axis**, x less than –2.71, x between 1 and 2.21, and x greater than 4: (-∞, -2.71] or (1, 2.21] or (4, ∞). The **dots required brackets**.