# Math 110 Lecture II: Polynomials, Rational Functions and Logarithms

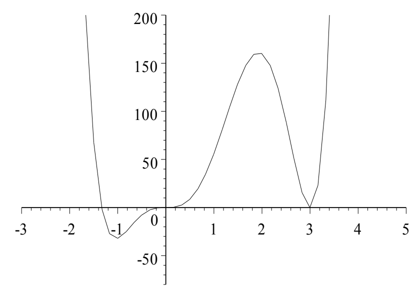
A) Polynomials are like functions having only addition, subtraction, and multiplication with variables (division and taking roots are prohibited). Examples: $P(x) = 3x^3 + 4x^2 - 5x - 10$; $P(x) = 6$; $P(x) = 4 - x^2$; $P(x) = 2x^3(3 - x)^2(3x + 4)$.

1) Every polynomial has **degree** (or order) and some can be named by the number of terms.
   a) The **degree** of a polynomial is the **highest power in any term**, or **the number of variable factors** if in factor form. For the examples above the 1st has degree 3, the 2nd degree 0, the 3rd degree 2, and the 4th degree 6.
   b) A polynomial with **one** term is called a **monomial**; **two** terms a **binomial**; and **three** terms a **trinomial**.

2) A polynomial is in **term form** when **+ and –** signs separate the various parts.
   a) The **coefficient** of each term is the number factor and includes the sign. The **Leading coefficient** is the coefficient of the highest power term. The **leading coefficient** for the 1st is 3, the 2nd is 6, the 3rd is –1, and for the 4th you must imagine all the factors multiplied together – it is 6.
   b) The chief **advantage** of term form is the immediate use of the '**poly**' function on the TI 85 and 86 to find the roots (x-intercepts) for the polynomial.

3) A polynomial is in **factor** form when '**multiplication**' separates the various parts like in the 4th polynomial above.
   a) The 4th polynomial above could be written $P(x) = 2\ x\ x\ x\ (3 - x)\ (3 - x)\ (3x + 4)$, there are 7 factors but only 6 with a variable so the degree is 6.
   b) The advantage of factor form is in graphing because the **roots** (x-intercepts) can be obtained **by setting each variable factor to zero and solving**, so 0 is a root three times, 3 is a root twice and –1.333 is a root once. The behavior of the graph at the roots is determined by the number of times (**multiplicity**) a particular number is a root:
   -if **once** the graph passes through the x-axis **undisturbed**
   -if **twice** (or any even number of times) the graph will **kiss** the x-axis
   -if **three** times (or other odd) the graph will **french kiss** the x-axis (look at the example to see what this means, from the right the graph seems to be going to kiss the axis at 0 but then it crosses through and falls)

   For $P(x) = 2x^3(3 - x)^2(3x + 4)$, the graph will cross the x-axis undisturbed at –1.333, it will kiss the x-axis at 3, and it will french kiss at 0.




4) A polynomial in term form may also be written using the **Remainder Theorem**: $P(x) = (x - a)\,Q(x) + R(x)$, where Q(x) is the quotient, R(x) is the remainder and (x – a) is a factor with root 'a'. Since (x – a) is linear Q(x) can be obtained using Synthetic Division.
   Examples: If a = -2, the 1st and 3rd polynomials above are first divided synthetically by –2 as follows:

```
-2|  3   4   -5   -10          -2|  -1   0    4
     0  -6    4     2                0   2   -4
     3  -2   -1  | -8               -1   2  | 0
```

   So $3x^3 + 4x^2 - 5x - 10 = (x+2)\,(3x^2 - 2x - 1) - 8$ and $4 - x^2 = (x + 2)\,(-x + 2) + 0$

B) Rational Functions are fractions with a polynomial in the numerator and a polynomial in the denominator they can be graphed quickly by following the 5 steps listed below:

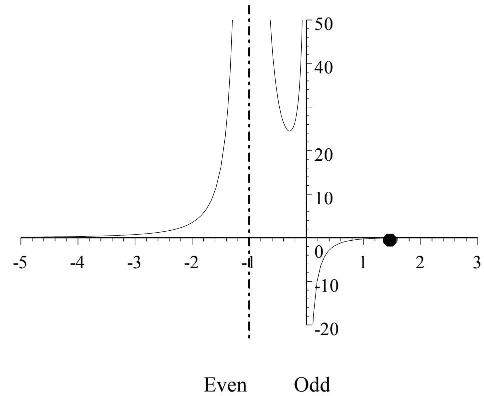
1) **Factor** the numerator and the denominator **and reduce** if possible. 'Poly' can be used if desired, if the same root is in the numerator and the denominator canceling them is the same as reducing.
2) The **Real roots** of the **numerator** are the only **x-intercepts**, graph them (remember multiplicity).
3) The **Real roots** of the **denominator** are **vertical asymptotes**, graph them as vertical dotted lines, if they occur an odd number of times call the vertical asymptote **ODD** and the graph goes to opposite ends on each side, if an even number of times call it **EVEN** and the graph goes to the same end on opposite sides.
4) **Compare** the **degree** of the numerator, N, with the **degree** of the denominator, D:
   If **N < D**, there is a **horizontal asymptote** at **y = 0**, since for large values of x the denominator would be much larger than the numerator. To see if the graph is above the line y = 0 (x-axis) or below on the far right, consider if y is positive for a large x or negative (dividing the leading coefficients will determine this).

   If **N = D**, there is a **horizontal asymptote** at $y = \frac{a}{b}$, where **a and b** are the **leading coefficients** respectively.

   If **N > D**, there is an **oblique asymptote** at **y = Q(x)**, Q(x) is the quotient obtained from **long division**.
5) To validate your graph select another point or two (it is always best to begin graphing from the right).

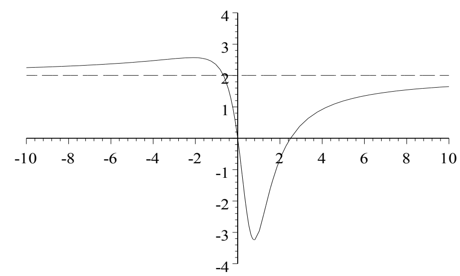
Example 1: $y = \dfrac{2x^2 - 3x}{x^2 + 2x^3 + x^4}$; 1) factoring gives $y = \dfrac{x(2x-3)}{x\ x(1+x)(1+x)}$ and reducing gives $y = \dfrac{(2x-3)}{x\,(1+x)(1+x)}$

2) The only real root of the numerator is 1.5, so it is the only x-intercept.
3) The real roots of the denominator are 0, and 1 twice, so there is an ODD vertical asymptote at 0 and an EVEN one at 1.
4) The degree on top is 2 and on the bottom is 4, N<D, so a horizontal asymptote exists at y = 0, and for a large x value it is positive so the graph is above the x-axis on the right.
5) Other points could be chosen but none are necessary.




Example 2: $y = \dfrac{2x^3 - 3x^2 - 5x}{x^3 + 1}$; 1) using 'poly' the roots are: $\dfrac{0 \quad 2.5 \quad \cancel{-1}}{\cancel{-1} \quad (.5, .87)(.5, -.87)}$, cancel the –1

2) The real roots in numerator are: 0 and 2.5, so these are the only x-intercepts.
3) There are no real roots left in the denominator, so there are no vertical asymptotes.
4) The degrees are the same so N = D, therefore the horizontal asymptote is $y = \dfrac{2}{1} = 2$ (leading coefficient over leading coefficient).
5) The graph starts at the horizontal asymptote on the far right so no other points are necessary.

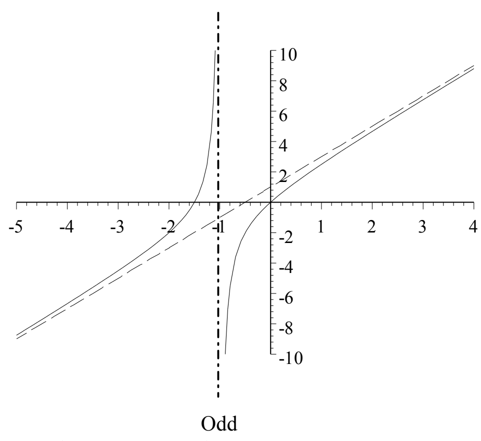



Example 3: $y = \dfrac{2x^3 + x^2 - 3x}{x^2 - 1}$; 1) factoring gives $y = \dfrac{x(2x+3)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$

2) The real roots in numerator are: 0 and –1.5, so they are the only x-intercepts
3) The only real root left in the denominator is –1, so the vertical asymptote at –1 is ODD.
4) The degree on top is 3, on bottom is 2 so, N > D. Therefore there is an oblique asymptote at y = Q(x), or in this case at

y = 2x + 1, since:

$$\begin{array}{r|l} & 2x + 1 \\ \hline x^2 - 1 & 2x^3 + x^2 - 3x \\ & \underline{-2x^3 \qquad + 2x} \\ & x^2 - x \\ & \underline{-x^2 \qquad + 1} \end{array}$$




C) Logarithms – a new country with new laws, rules and standards, pay attention to these NEW rules.

1) **Rules** that must be learned and understood for this new country.

a) **Definition**: **If $y = b^x$ then $x = \log_b(y)$**; examples: $6^{2-x} = 11 \longleftrightarrow 2 - x = \log_6(11)$; $\log_2(3x-4) = 4 \longleftrightarrow 2^4 = 3x - 4$; $10^x = 15 \longleftrightarrow x = \mathbf{\log_{10}}(15)$, however when the base is 10 we do not write it, so $x = \mathbf{\log}(15)$; similarly when the base is 'e', like in $\mathbf{\log_e}(x) = 4$, we write $\mathbf{\ln}(x) = 4$ instead, so if $e^{2x} = 5 \longleftrightarrow \ln(5) = 2x$; The double arrow means '**by definition**'.

b) **Identities** (4 and they come from the definition applied to: $b^0 = 1$; $b^1 = b$; $b^n = b^n$; and $\log_b(x) = \log_b(x)$):

i) $\log_b(1) = 0$ examples: $\log_2(1) = 0$; $\log(1) = 0$; $\ln(1) = 0$; $\log_8(1) = 0$

ii) $\log_b(b) = 1$ examples: $\log_2(2) = 1$; $\log(10) = 1$; $\ln(e) = 1$; $\log_6(6) = 1$

iii) $\log_b(b^n) = n$ examples: $\log_2(8) = 3$; $\log(100) = 2$; $\ln(e^{-1}) = -1$; $\log_{25}(5) = .5$

iv) $b^{\log_b(x)} = x$ examples: $2^{\log_2(x)} = x$; $10^{\log(4)} = 4$; $e^{\ln(x)} = x$; $3^{\log_3(2x+1)} = 2x + 1$

c) Laws (4 also)

i) $\log_b (xy) = \log_b (x) + \log_b (y)$ examples: $\log_2 (3x) = \log_2 (3) + \log_2 (x)$; $\ln (12) = \ln (2) + \ln (6)$.

ii) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ examples: $\log\left(\frac{5}{12}\right) = \log(5) - \log(12)$; $\log_3\left(\frac{x}{6}\right) = \log_3(x) - \log_3(6)$.

iii) $\log_b (x^n) = n \log_b (x)$ examples: $\ln (x^4) = 4 \ln (x)$; $2 \log_5 (y) = \log_5 (y^2)$; $3 \log_2 (5) = \log_2 (125)$.

iv) $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$ examples: $\log_4(6) = \frac{\log(6)}{\log(4)}$; $\log_8(4) = \frac{\log_2(4)}{\log_2(8)} = \frac{2}{3}$; $\log_5(20) = \frac{\ln(20)}{\ln(5)}$

2) **Applications** of logarithms:

a) **Solving equations** that have logarithms or exponentials can best be done by:
1st Using the definition if possible (to get rid of the 'log' or to solve for the power by adding a 'log'), or
2nd Using an 'identity' or a 'law' so the definition can be used, or
3rd Using 'algebra' to change the form so either of the first two steps can be done.
Example 1: $3^{2x+1} = 4^{3x}$; using the definition gives, $\log_3 (4^{3x}) = 2x + 1$, but now using law iii on the left gives, $3x \log_3 (4) = 2x + 1$, which simplifies to $3x (1.262)$ from law iv; therefore $3.786x = 2x + 1$, so $1.786x = 1$ and finally $x = .56$. It is always a good idea to check your answer: $3^{(1.12+1)} = 3^{2.12} = 10.268 = 4^{1.68}$, so it checks.
Example 2: $3 - \log_c (x) = \log_2 (x - 2)$; Using algebra gives, $3 = \log_2 (x - 2) + \log_2 (x)$, but using law i simplifies to, $3 = \log_2 (x^2 - 2x)$, and now the definition gives, $2^3 = x^2 - 2x$ or $x^2 - 2x - 8 = 0$, which factors into $(x - 4)(x + 2) = 0$ giving the solutions $x = 4$, and $x = -2$. Checking 4 gives $1 = 1$, but checking $-2$ gives $3 - \log_2 (-2) = \log_2 (-4)$ and since logs of negatives give complex answers $-2$ is not a solution; therefore the only solution for this equation is $x = 4$.

b) **Exponential growth and decay** problems that use the formula: $\mathbf{A = P e^{kt}}$, A = ending amount, P = beginning amount, k = the rate of growth or decay, and t = time.
Example 1: When would you predict there will be 50 million members of the church if the 1st million was reached in 1947 and in 2001 the membership reached 11 million?
These problems always have **two parts**, **part 1** requires us to find the **rate of growth** from the given data: A=11 million, P=1 million, and time is the difference or 54 years, so $11 = 1 e^{54k}$; use algebra to divide both sides by 1 getting, $11 = e^{54k}$, the definition now gives $54 k = \ln (11)$, so $k = .044$. **Part 2** uses the result for k, P = 1 and A = 50 to solve for t: $50 = 1 e^{.044 t}$, therefore solving for t gives, $.044 t = \ln (50)$, or $t = 88.91$. If we add 88.91 years to 1947 we could expect 50 million members in the latter end of 2035.

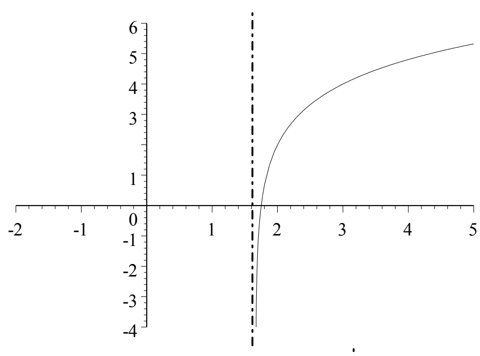
c) Graphing logarithm functions of the type: $y = \log_2 (3x - 5) + 2$, requires following the steps:

i) Find the y-intercept by letting $x = 0$, for our example $y = \log_2 (-5) + 2$, but the log of a negative is not real so there is no y-intercept.

ii) Find the vertical asymptote by finding the root of the expression being logged, $3x - 5 = 0$, $x = 1.67$ (there is always a vertical asymptote but never a horizontal one for logs)

iii) Find the x-intercept by letting $y = 0$ and solving for x: $0 = \log_2 (3x - 5) + 2$, subtract 2 from both sides getting, $-2 = \log_2 (3x - 5)$, use the definition to get $2^{-2} = 3x - 5$, so $.25 = 3x - 5$, $3x = 5.25$, $x = 1.75$

iv) Find another point by selecting a value for x that will give a positive number in the log, 2 or bigger: if $x = 2$ then $y = \log_2 (6-5) + 2 = \log_2 (1) + 2 = 2$, so another point is (2,2).




Example 2: $y = 1 - \log_3 (3 - 2x)$

i) y-intercept is $y = 1 - \log_3 (3) = 1 - 1 = 0$

ii) The vertical asymptote is when $3 - 2x = 0$ or $x = 1.5$

iii) The x-intercept is at 0 too, since the y-intercept is at 0.

iv) Another point is possible for any x value left of 1.5, if $x = -3$, $y = 1 - \log_3 (3 - 2(-3)) = 1 - \log_3 (3 + 6) = 1 - 2 = -1$, so the point is (-3, -1).