Math 110 – Lecture III: Matrices, Systems of Equations, Conic Sections and The Binomial Series.

A) Matrices are arrays of numbers in rows (n) and columns (m). The dimension of a matrix is written: n×m, and expresses the size of the matrix.

1) Matrices can be **added** and **subtracted** if only if they have the **same dimensions**:

$$\begin{bmatrix} 2 & -1 & 4 & 6 \\ -3 & 0 & 5 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 5 & -4 & 0 \\ 2 & 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 & 3 & -1 \\ -4 & 1 & 2 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 6 & -3 & 7 \\ 3 & -1 & 1 & -4 \end{bmatrix}$$ Notice corresponding elements were added and subtracted for each position.

2) **Identity** matrices are square matrices with **1's** down the **major diagonal** (top left to bottom right) with all **other elements** being **zero**: 1×1: $\begin{bmatrix} 1 \end{bmatrix}$ 2×2: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 3×3: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

3) Matrices can be **multiplied** if only if the number of **columns of the first** matrix is exactly the **same as the number of rows of the second** matrix and the product matrix will have the **dimension**: rows of 1^{st} × columns of 2^{nd}. Each element of the product comes from the product of corresponding elements of a row of the 1^{st} times corresponding elements of a column of the 2^{nd} and the results added.

Can these be multiplied? $\begin{bmatrix} 1 & -3 & 2 & 4 \\ -2 & 0 & 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix}$ The 1^{st} has dimension: 2×**4**, the 2^{nd} dimension: **4**×1, so they can be multiplied and the product will have dimension: 2×1. Each element in the product will be a row of the 1^{st} times corresponding elements of the 2^{nd} and then added: -2 + -9 + -2 + 16 = 3, and 4 + 0 +-3 + 32 = 33; so the product is $\begin{bmatrix} 3 \\ 33 \end{bmatrix}$.

4) The **Inverse** of a matrix only exists if the matrix is **square** and then there are some matrices that do not have inverses [Finding out how to tell which ones do not have inverses is an excellent homework challenge question]. Inverses are found using the '**x^{-1}**' key on your calculator.

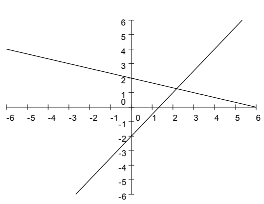
$$\begin{bmatrix} -1 & 4 \\ 5 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} .111 & .222 \\ .278 & .056 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & 2 \\ -4 & 0 & -2 \\ 7 & -10 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} -.286 & -.543 & -.086 \\ .143 & -.029 & -.057 \\ .57 & .586 & .171 \end{bmatrix}$$

Note: **Multiply** the **original** matrix **by** its **inverse** and see what you get (whenever you have numbers with E-9 or E-14, the calculator should have put 0). The result should have been the **Identity** matrix in each case.

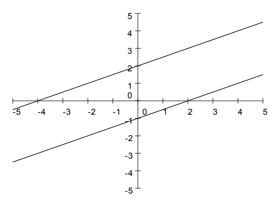
5) Division is not possible with matrices but it is possible to multiply by the Inverse. The **product** of a **matrix** and **its inverse** is always the **identity** matrix.
Matrices can be identified with capital letters: I = the identity, A^{-1} = the inverse of matrix A, X = variable matrix.
$A \cdot X = B$ is therefore a matrix equation and since division is not possible, the only way to solve is to multiply both sides by A^{-1}, but order matters for matrix multiplication so we must multiply on the left for both sides: $A^{-1}A \cdot X = A^{-1}B$. $A^{-1}A$ though is equal to I so: $I \cdot X = A^{-1}B$, and since $I \cdot X = X$, the solution is: $X = A^{-1}B$.

B) Systems of equations generally have as many equations as unknowns. They were solved in earlier math courses using three different techniques: **graphing**, **substitution** and **addition-subtraction**; these methods may still be used but a better and faster method will be presented in step 2.

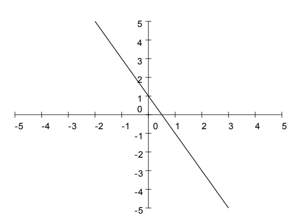
1) First, however, it is important to realize that all systems of equations will have one of **three possible solution sets**, and these three possibilities are illustrated in the way two straight lines can be graphed on the same axis:



They intersect – **one** point is **solution**
Consistent and independent



They are parallel – **no solution**
Inconsistent and independent



They are the same line – **infinite solutions**
Consistent and dependent

2) We solve systems of equations with a new method that uses Matrices. The method has **three** names: The **Augmented Matrix Method**, Row-Equivalent Eschelon Form (rref), and Guassian Elimination; we will use the first name the most. The steps follow:

a) From the system **write an augmented matrix** (always it must have **one more column** than it has rows). The system must first be written with all variable terms in the left member and a constant only in the right member. Put this matrix in your calculator.

b) Now '**rref**' your matrix.

c) Write your resulting matrix **back into equations** (reverse step one).

d) **Interpret your solution** by looking at the equations in the third step; i.e. x = 2, y = -1, z = 4 would give the single point (2, -1, 4) as the solution; x = 4, y – 2z = 3, and 0 = 1 would give '**no solution**' since 0 = 1 is a **LIE**; x = 4, y – 2z = 3, and **0 = 0** would give '**infinite solutions**', three are: (4, 5, 1), (4, 7, 2) and (4, 1, -1).

Example: $\begin{array}{r} 2x-3y-z=0 \\ x+2y+2z=3 \\ -x-2y+4z=5 \end{array} \Rightarrow \begin{bmatrix} 2 & -3 & -1 & 0 \\ 1 & 2 & 2 & 3 \\ -1 & -2 & 4 & 5 \end{bmatrix} \mathit{rref} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & .52 \\ 0 & 1 & 0 & -.10 \\ 0 & 0 & 1 & 1.33 \end{bmatrix} \Rightarrow \begin{array}{l} x=.52 \\ y=-.10 \\ z=1.33 \end{array}$ solution is (.52, -.1, 1.33)

$\begin{array}{r} 2x-y-z=-3 \\ x-4y+2z=3 \\ -4x+2y+2z=6 \end{array} \Rightarrow \begin{bmatrix} 2 & -1 & -1 & -3 \\ 1 & -4 & 2 & 3 \\ -4 & 2 & 2 & 6 \end{bmatrix} \mathit{rref} \Rightarrow \begin{bmatrix} 1 & 0 & -.86 & -2.14 \\ 0 & 1 & -.71 & -1.29 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \mathbf{x-.86z=-2.14} \\ \mathbf{y-.71z=-1.29} \\ \mathit{0=0} \end{array}$ since 0 = 0 there are infinite solutions and we select different ones by letting z be different numbers since z is in both equations: if z = 0 the solution is (-2.14, -1.29, 0), if z = 1 it is (-1.28, -.58, 1), if z = -1 it is (-3, -2, -1), etc.

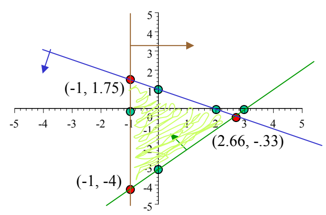
3) Solving **systems** of linear **inequalities** can only be done **by graphing** and each inequality in the system must have the **same two variables** and only two variables, since the graphing technique only applies to systems with two variables. Inequalities not only have the graph of a straight line but require that one side of the line be shaded too.

Systems of inequalities can have more than two inequalities

Solutions to these systems require the combined shaded region and all corner points

Find the x and y intercepts for each equations, graph and connect each pair; remember equation 3 gives a vertical line.

1) $x - y \le 3$
2) $2x + 4y \le 4$
3) $x \ge -1$



Equation 1: x = 3 and y = -3; plot and connect
(0, 0) makes it true so shade top side.

Equation 2: x = 2 and y = 1; plot and connect
(0, 0) makes it true so shade bottom side.

Use the augmented matrix method with equations 1 and 2 to find their intersection point.
Therefore the corner point where equations 1 & 2 meet is (2.66, -.33)

Equation 3: x = -1 and no y-intercept; plot and draw a vertical line
(0, 0) makes it true too, since 0 is bigger than –1, so shade to the right

Since (0, 0) made all three equations true the region common to all contains (0, 0)
and it has three corners: (2.66, -.33) the intersection of equations 1& 2, and since equation 3 has x = -1, it intersects equation 1 at the point (-1, -4) and equation 2 at the point (-1, 1.75)

Note: Inequalities combined with a conditional equation create a very useful application called **Linear Programming**. The **conditional equation** is needed to find the **maximum or minimum** of some variable. For instance, suppose it was determined that F = 10x – 3y (a conditional equation) and combined with the inequalities graphed above you were asked to maximize F. To find the maximum for F all that is required is to substitute the three corners in the shaded region above into the F equation getting (-1, 1.75) → -15.25, (-1, -4) → 2 and (2.66, -.33)→27.59. It is obvious that the maximum value is 27.59 therefore the solution is (2.66, -.33), or you could say that 2 and two thirds of variable x combined with a negative one third of variable y gives a maximum.

C) **Conic Sections** include the following figures: **Circle, Ellipse, Hyperbola** and **Parabola**. They are not functions except for parabolas that open up or down. The **general form** for a conic is: $\mathbf{Ax^2 + By^2 + Dx + Ey + F = 0}$ ('A' can always be assumed to be a positive number since you could multiply both members by a -1).

1) Examining and comparing A and B can tell you which conic a specific formula will give (when numbers replace the capital letters):

a) If **A = B** the conic will be a **circle**.

b) If both **A and B are positive but not equal** the conic will be an **ellipse**.

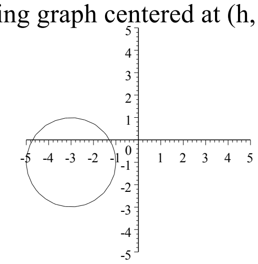
c) If **A or B is negative** the conic will be a **hyperbola**.

d) If **A or B is zero** the conic will be a **parabola**.

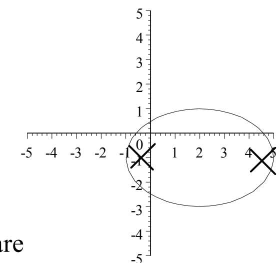
Examples: $3x^2 - 4x + 5 = 3y^2$, is a hyperbola, why? $2x - 3y + x^2 = 6 - y^2$, is a circle, why?

2) Specific Form for each conic, with the corresponding graph centered at (h, k):

a) **Circle** centered at (0, 0): $x^2 + y^2 = r^2$;
centered at (h, k): $(x - h)^2 + (y - k)^2 = r^2$

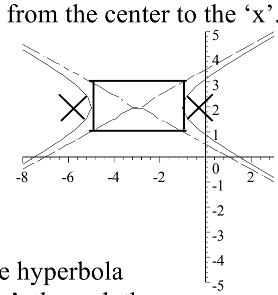


b) **Ellipse** centered at (0, 0): $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, a^2 can be under x^2 or y^2;



centered at (h, k): $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, there are also focal points **a distance 'c'** from the center on the larger axis (always 'a'). The relationship between a, b, and c for all ellipse is: $\mathbf{a^2 = b^2 + c^2}$. The foci are identified in the graph as x's. Note: 'a' is the distance from the center to the furtherest point of the ellipse, 'b' the distance from the center to the nearest point of the ellipse, and 'c' the distance from the center to the 'x'.

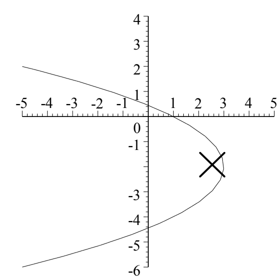
c) Hyperbola centered at (0, 0): $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, the '-' can be with x or y;



centered at (h, k): $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, there are also focal points

a distance 'c' from the center on the **positive** axis. The relationship between a, b, and c for all hyperbola is: $\mathbf{c^2 = b^2 + a^2}$. To construct the hyperbola a box is drawn around the center by going up 'b' and down 'b', left 'a' and right 'a', through the corners and center of the box two asymptotes are then drawn and then the foci are plotted 'c' units from the center on the positive axis, in the example the two pieces of the hyperbola touch the sides of the box near the foci.

d) Parabola with vertex at (0, 0): $4px = y^2$, the 'x' and 'y' can trade places;
vertex at (h, k): $4p(x - h) = (y - k)^2$, **'p'** is the distance between the **vertex** and the **focal point,** if negative the focal point is in the negative direction from the vertex (like in the graph to the right).



3) Changing from general form to specific form requires – identifying the conic, visualizing the goal (specific form) and completing the square to obtain the specific form:

Example 1: $3y^2 + 6x + 12y - 6 = 0$, the conic is a parabola since A = 0 (no x^2 term)

Our goal is to change the form into: $4p(x - h) = (y - k)^2$ by moving terms and completing the square

$-6x + 6 = 3y^2 + 12y$ we can divide each term by 3 to eliminate the coefficient in front of y^2,

$-2x + 2 = y^2 + 4y$ now complete the square by adding 4 to both members

$-2x + 2 + 4 = y^2 + 4y + 4$

$-2x + 6 = (y + 2)^2$ the left member now must be factored taking -2 out of both terms

$-2(x - 3) = (y + 2)^2$ The form now meets our goal if $4p = -2$, or $p = \dfrac{-1}{2}$, with (h, k) = (3, -2). The x-intercept is found by letting y = 0 in the original giving x = 1, the y-intercepts are found by letting x = 0 giving the quadratic: $3y^2 + 12y - 6 = 0$, which has roots at y = -4.45 and y = .45. Notice the graph of the parabola above is this parabola.

Example 2: $-2x^2 + 8y^2 - 12x - 32y + 22 = 0$, the conic is a hyperbola since A and B have different signs, therefore our goal is to get it in the form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ by completing the square in both x and y.

$-2x^2 - 12x \quad + 8y^2 - 32y \quad = -22$
$-2(x^2 + 6x + 9) + 8(y^2 - 4y + 4) = -22 - 18 + 32 = -8$
$-2(x+3)^2 + 8(y-2)^2 = -8$ and now divide each term by -8 getting:

$\frac{-2(x+3)^2}{-8} + \frac{8(y-2)^2}{-8} = 1$ reducing and paying attention to the negatives gives:

$\frac{(x+3)^2}{4} - \frac{(y-2)^2}{1} = 1$ so the center is (-3, 2), 'a' = 2 and 'b' = 1, with the positive axis 'x'. So from the center you go up 1 unit, down 1 unit, left 2 units and right 2 units, draw the box, then draw oblique asymptotes through the corners and center, and draw the two arcs so they touch the box on the sides and extend toward the two asymptotes. Furthermore, since $c^2 = a^2 + b^2$ for hyperbolas, $c = \sqrt{5} = 2.24$, and the two foci are drawn 2.24 units from the center inside the two arcs. The graph is given as the hyperbola above.

D) **Series** including the Binomial Series are sums with a pattern.

1) Often **series** are represented using **Summation Notation**, which includes the summation symbol, $\sum$, with a formula and limits with the symbol giving the beginning integer below and the ending integer above. The sum requires each term added beginning with bottom integer substituted into the formula and every integer inturn being substituted until the last integer on top is substituted.

Examples: $\sum_{k=2}^{5} k^2 = 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 = 54$, $\sum_{k=-1}^{1} \frac{k}{k+3} = \frac{-1}{2} + \frac{0}{3} + \frac{1}{4} = \frac{-1}{4}$

Two special types: **Arithmetic** (terms have a common difference) and **Geometric** (terms have a common ratio).

2) Series require the use of **Factorials**, written: n!. $n! = n\cdot(n-1)\cdot(n-2)\cdots 1$; $2! = 2\cdot 1 = 2$; $5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$.
0! is defined as 1. **Combinations**: The number of combinations of n objects taken k at a time, ${}_nC_k = \frac{n!}{(n-k)!k!}$,

Example: Suppose you were a basketball coach and had 10 players, how many ways could you take 5 at a time and play them on the court? ${}_{10}C_5 = \frac{10!}{(10-5)!5!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{5\cdot 4\cdot 3\cdot 2\cdot 1\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 9\cdot 2\cdot 7\cdot 2 = 252$

3) Pascal's Triangle

Each row of numbers comes from the coefficients obtained when (a+b) is raised to a power, the first row is $(a+b)^0$ which is 1, next $(a+b)^1$ giving 1 and 1, then $(a+b)^2$ giving $a^2+2ab+b^2$, or 1, 2, & 1, and so forth. Can you see how each row of numbers can be generated from the numbers in the row above them?

	Sum
1	1
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64
1 7 21 35 35 21 7 1	128

The bottom row represents: $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$; Notice the two 35's, they are associated with powers on a and b of 3 and 4, what is ${}_7C_3$?

$\frac{7!}{(7-3)!3!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1} = 7\cdot 5 = 35$, in fact each of the individual numbers is a combination.

4) **The Binomial Theorem**: Now combining these several concepts gives the following formula:

$(a+b)^n = \sum_{k=0}^{n} {}_nC_k a^{n-k} b^k$; the binomial theorem is used the following two ways:

To expand anything like $(3-2x)^7$: since $(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$, if a is replaced by '3' and b by '-2x', $(3-2x)^7 = 3^7 + 7\cdot 3^6(-2x) + 21\cdot 3^5(-2x)^2 + 35\cdot 3^4(-2x)^3 + 35\cdot 3^3(-2x)^4 + 21\cdot 3^2(-2x)^5 + 7\cdot 3(-2x)^6 + (-2x)^7$. So $(3-2x)^7 = 2187 - 10{,}206x + 20{,}412x^2 - 22{,}680x^3 + 15{,}120x^4 - 6{,}048x^5 + 1344x^6 - 128x^7$.

To find a single term, like the 21st term in $(2-x)^{22}$: k is 20, so 21st term is ${}_{22}C_{20}\cdot 2^2\cdot(-x)^{20} = 231\cdot 4\cdot x^{20} = 924x^{20}$.