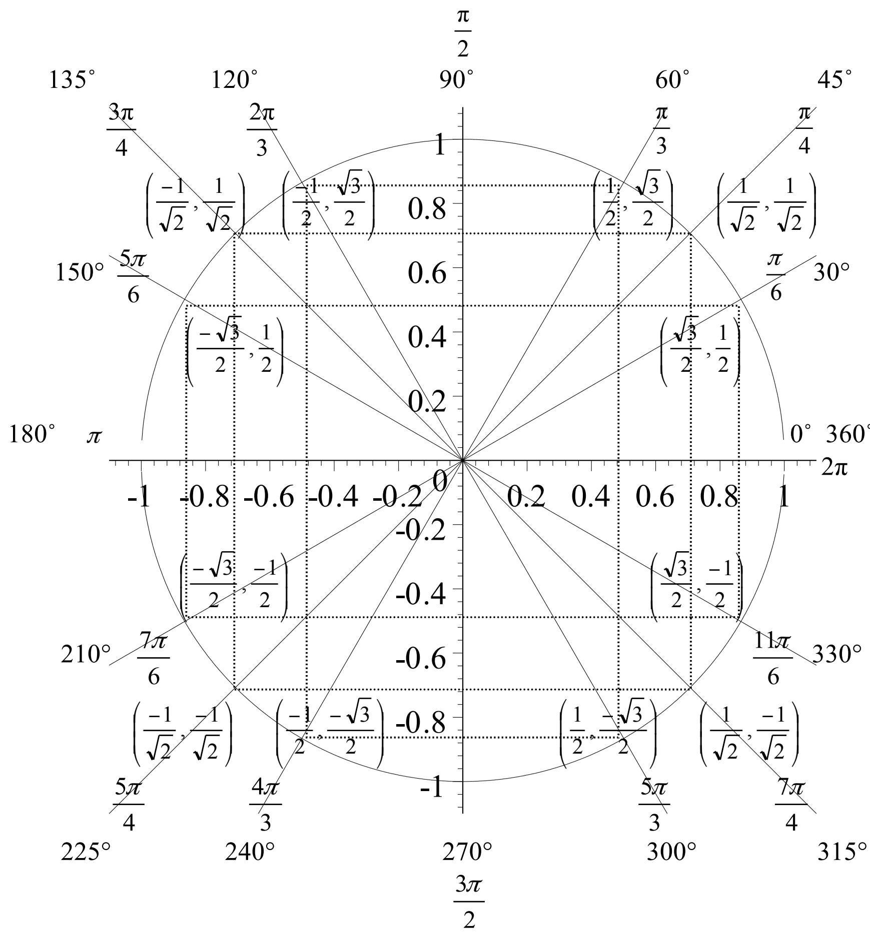
Math 111 Lecture I – The Unit Circle, The Six Trig Functions Defined and Graphed and Their Inverses Defined and Graphed, also the Magnificent Seven.

A) Trigonometry Defined using the Unit Circle

1) The Unit Circle with 16 points and 17 angles + infinity

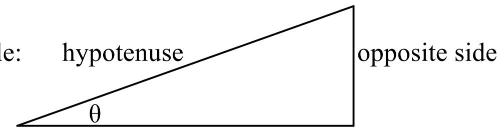


Notice all points at the $\frac{\pi}{6}$ angles have the very same x and y except of course for signs, and the points at $\frac{\pi}{3}$ angles just have the x and y switched. Of course all $\frac{\pi}{4}$ angles have the same x and y. Notice also that all $\frac{\pi}{6}$ angles are closer to the x-axis while all $\frac{\pi}{3}$ angles are closer to the y-axis. Each quadrant has 3 angles in the same relative positions. Also we could have gone clockwise rather than counterclockwise and named every angle with a negative name; i.e. $\frac{11\pi}{6}$ would be named $\frac{-\pi}{6}$ and so forth. This last comment suggests that every angle could be named two ways but the truth is that every angle can be named infinitely many ways because we can go around again and again; i.e. $\frac{11\pi}{6} = \frac{-\pi}{6} = \frac{23\pi}{6} = \frac{-13\pi}{6}$ you just keep adding 2π or in sixths, $\frac{12\pi}{6}$.

2) There are six trig functions and they are defined using the x and y from the points on the unit circle as follows (with **reciprocal functions paired**, i.e. sin θ is the reciprocal of csc θ, etc.):

		Example for $\frac{2\pi}{3}$:	
a) $\sin\theta = y$	f) $\csc\theta = \frac{1}{y}$	a) $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$	f) $\csc\frac{2\pi}{3} = \frac{2}{\sqrt{3}}$
b) $\cos\theta = x$	e) $\sec\theta = \frac{1}{x}$	b) $\cos\frac{2\pi}{3} = \frac{-1}{2}$	e) $\sec\frac{2\pi}{3} = -2$
c) $\tan\theta = \frac{y}{x}$	d) $\cot\theta = \frac{x}{y}$	c) $\tan\frac{2\pi}{3} = -\sqrt{3}$	d) $\cot\frac{2\pi}{3} = \frac{-1}{\sqrt{3}}$

Note: There are definitions for trigonometry also using a right triangle: Only the first three functions will be shown but the others are still reciprocals of the first three:



$\sin\theta = \dfrac{\text{opp}}{\text{hyp}}$ $\qquad$ $\cos\theta = \dfrac{\text{adj}}{\text{hyp}}$ $\qquad$ $\tan\theta = \dfrac{\text{opp}}{\text{adj}}$ $\qquad$ [<u>O</u>h <u>H</u>eck <u>A</u>nother <u>H</u>our <u>O</u>f <u>A</u>lgebra]

3) Using the definitions of part two and the Unit Circle of part one it is possible to construct the following table (this is called the **Restricted Table**): The first table uses Unit Circle fractions, the second table uses decimal equivalents:

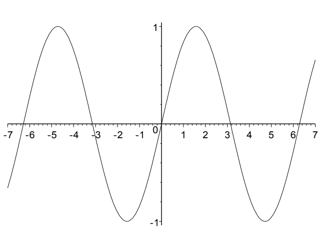
θ	$\frac{-\pi}{2}$	$\frac{-\pi}{3}$	$\frac{-\pi}{4}$	$\frac{-\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Sin θ	-1	$\frac{-\sqrt{3}}{2}$	$\frac{-1}{\sqrt{2}}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1				
Cos θ					1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-1}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}$	-1
Tan θ	und	$-\sqrt{3}$	-1	$\frac{-1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	und				
Cot θ					und	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$\frac{-1}{\sqrt{3}}$	-1	$-\sqrt{3}$	und
Sec θ					1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	und	-2	$-\sqrt{2}$	$\frac{-2}{\sqrt{3}}$	-1
Csc θ	-1	$\frac{-2}{\sqrt{3}}$	$-\sqrt{2}$	-2	und	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1				

θ	-1.57	-1.05	-.79	-.52	0	.52	.79	1.05	1.57	2.09	2.36	2.62	3.14
Sin θ	-1	-.87	-.71	-.5	0	.5	.71	.87	1				
Cos θ					1	.87	.71	.5	0	-.5	-.71	-.87	-1
Tan θ	Und	-1.73	-1	-.58	0	.58	1	1.73	und				
Cot θ					Und	1.73	1	.58	0	-.58	-1	-1.73	und
Sec θ					1	1.16	1.41	2	und	-2	-1.41	-1.16	-1
Csc θ	-1	-1.16	-1.41	-2	und	2	1.41	1.16	1				

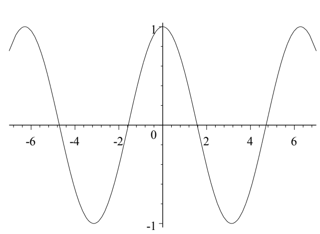
We could have filled in every box but doing so gives functions that are not one-to-one. The purpose of this table is to change the six trig functions to **one-to-one functions** so they can have inverses. For **one** of your **homework** problems prepare a table of 21 angles beginning with $-\pi/2$ (use that table to graph the functions in part 4).

4) From the homework problem table it is possible to graph each of the six trig functions by letting the x values be the angles from the top row of the table and the numbers in the function row be the y values respectively. The graphs follow (for the graphs all the fractions from the table are changed to decimal numbers, including the angles):

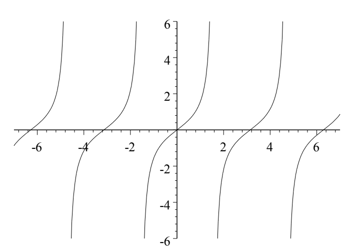
y = sin x



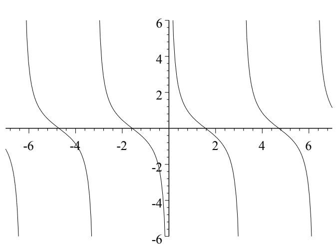
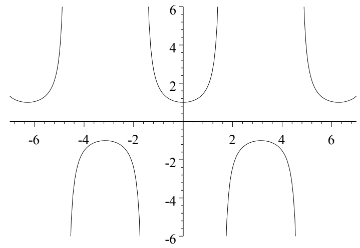
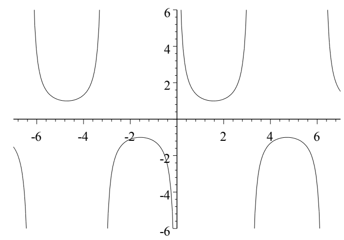
y = cos x



y = tan x



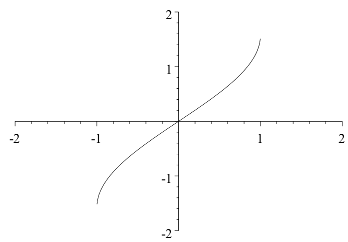
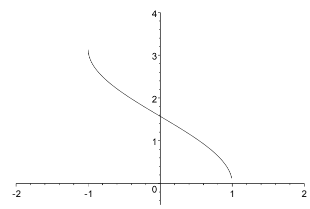
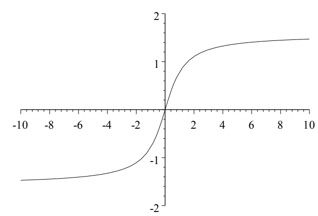
$y = \cot x$ $\qquad\qquad$ $y = \sec x$ $\qquad\qquad$ $y = \csc x$

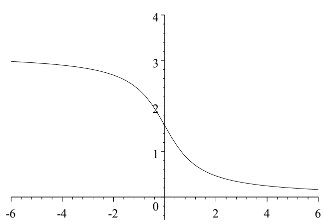
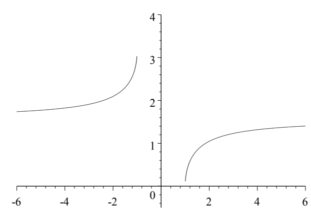
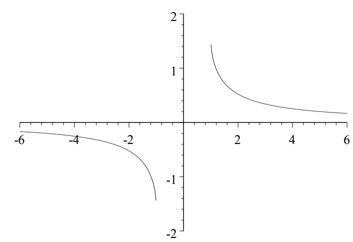
Notice these graphs repeat, the repeating **period** for all functions except tan x and cot x is **2π** (or in decimals 6.28). The repeating **period** for **tan x and cot x** is just **π** (3.14). Because these graphs repeat **they are not one-to-one** and therefore as they stand they do not have inverses. Also notice 'und' always translates into a vertical asymptote.

5) Using the Restricted table in part 3 and switching the x and y (meaning the y values now are the angles and the x-values are opposite the function name) we are able to graph the Inverse Trig functions given below:

$y = \sin^{-1} x$ ($y = \arcsin x$) $\qquad\qquad$ $y = \cos^{-1} x$ $\qquad\qquad$ $y = \tan^{-1} x$

$y = \cot^{-1} x$ $\qquad\qquad$ $y = \sec^{-1} x$ $\qquad\qquad$ $y = \csc^{-1} x$

Notice these functions are one-to-one, they do not repeat, for every x-value there is only one y-value and visa-versa.

6) Applications:

a) Evaluating function values for any angle and Inverse function values for any number:

i) Functions: For any unit circle angle use the large table – $\cos\left(\frac{7\pi}{6}\right) =$ $\qquad$ $\cot\left(\frac{7\pi}{4}\right) =$

$\csc\left(\frac{11\pi}{6}\right) =$ $\qquad$ $\tan\left(\frac{7\pi}{6}\right) =$ $\qquad$ $\sin\left(\frac{5\pi}{3}\right) =$ $\qquad$ $\sec\left(\frac{7\pi}{6}\right) =$

For radian angles not in unit circle – sin(.678) = $\qquad$ sec(5.66) = $\qquad$ cot(-3.45) =
(reciprocal functions must be used as needed)

ii) Inverse functions: For unit circle results use the restricted table, otherwise reflect on the following procedure: Suppose $\cot^{-1}(x) = y$ it is possible to remove the cot inverse by taking the cot of both sides

$$\cot[\cot^{-1}(x)] = \cot(y)$$

$x = \cot(y)$ but $\cot(y)$ is equal to $\dfrac{1}{\tan(y)}$ so $x = \dfrac{1}{\tan(y)}$ and this can be solved

for tan(y) getting: $\tan(y) = \dfrac{1}{x}$ and it is possible to solve for y by taking the inverse tangent of both sides

$$\tan^{-1}[\tan(y)] = \tan^{-1}\left(\frac{1}{x}\right) \text{ and so}$$

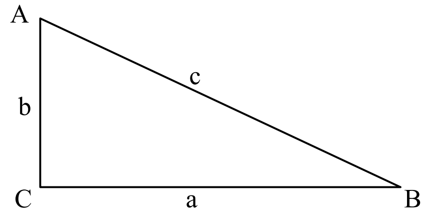
$$y = \tan^{-1}\left(\frac{1}{x}\right) \text{ but it also was equal to } \cot^{-1}(x) \text{ so } \cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$$

Note: The same thing can be done with: $\sec^{-1}(x)$ and $\csc^{-1}(x)$. There **is a problem** that occurs for the **inverse cotangent** function **for negative** numbers. Identify the problem and give an explanation for one **homework problem**.

$\cos^{-1}(-1) =$ $\qquad$ $\sin^{-1}\left(\frac{-\sqrt{3}}{2}\right) =$ $\qquad$ $\cot^{-1}(-4) =$ $\qquad$ $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) =$

$\csc^{-1}(-2) =$ $\qquad$ $\sec^{-1}(-4.3) =$ $\qquad$ $\csc^{-1}(10) =$ $\qquad$ $\cot^{-1}\left(-\sqrt{3}\right) =$

b) Applications using Right triangles:



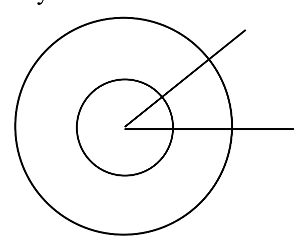
Suppose you are given a right triangle with a hypotenuse, c of 10 feet, with angle A, 50˚. What is the measure of the two legs and of angle B?

Since all three angles add up to 180˚ angle B must be 40˚.

Also since $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\sin 50^\circ = \frac{a}{10}$; therefore $a = 10 \sin 50^\circ = 7.66$ feet

Also $\cos 50^\circ = \frac{b}{10}$, therefore $b = 10 \cos 50^\circ = 6.43$ feet. A simple check is using the Pythagorean theorem, $a^2 + b^2 = c^2$ or $6.43^2 + 7.66^2 = 10^2 = 100 = 41.32 + 58.68$.

c) There are two velocities that come together in trigonometry they are **linear** and **angular** velocity.
The figure illustrates the concepts involved: If the small circle is the unit circle, radius 1, and the large circle has radius 'r', and if the angle is θ, and arc length, s, then the following ratio:



$\frac{\theta}{2\pi} = \frac{s}{2\pi r}$ exists, θ is to 2π (the full circle angle) as 's' is to the circumference of the big circle, 2πr; and solving for s gives: length of arc, **s = θr** , **all angles** must be in **radian** form. Now for the velocities:

-Linear velocity, **v**, is how fast you are going in a straight line, the **units** will always be **mph**, or **ft per second**.
The formula is: $v = \frac{\text{distance}}{\text{time}} = \frac{\theta r}{t}$

-Angular velocity, **ω**, is how fast you are spinning around, the **units** will always be **radians per minute**, or **revolutions per minute**, rpm. The formula is: $\omega = \frac{\text{angle}}{\text{time}} = \frac{\theta}{t}$

Note: Since $v = \frac{\theta r}{t}$ and $\omega = \frac{\theta}{t}$, we can write $v = \frac{\theta r}{t} = \frac{\theta}{t} \cdot r = \omega r$, so both velocities are related: **v = ω r**

Application: If you are riding a bicycle down the highway at an average speed of 32 mph, how fast is each tire spinning in rpm's if the tires are 26" diameter?
Solution: Since the diameter is 26", the radius is 13" and the speed is mph so 32 is linear velocity, therefore:

$$\omega = \frac{v}{r} = \frac{32\text{ miles}}{13\text{ inches}\cdot\text{hours}} \cdot \frac{1\text{ hr}}{60\text{ min}} \cdot \frac{5280\text{ ft}}{1\text{ mi}} \cdot \frac{12\text{ in}}{1\text{ ft}} \cdot \frac{1\text{ rev}}{2\pi}$$, therefore $\omega = 413.71$ rpm

Note: The units are the key, changing units requires canceling out all units other than the units needed for the desired velocity.

B) Identities: Equations that are true for all angles giving real results.
1) The Magnificent Seven Identities – all the Trig Identities needed for Calculus:
a) $\mathbf{\cos^2 x + \sin^2 x = 1}$ or by dividing by $\cos^2 x$: $\mathbf{1 + \tan^2 x = \sec^2 x}$ or by $\sin^2 x$: $\mathbf{\cot^2 x + 1 = \csc^2 x}$. This identity comes from the equation of the unit circle: $x^2 + y^2 = 1$, with x being replaced by cosθ and y being replaced with sinθ. Of course the θ can be replaced with any variable whatever.

b) **$\cos(x + y) = \cos x \cos y - \sin x \sin y$** called the sum formula for cosines. This identity comes from the fact that the distance between the following two sets of points on the unit circle is equal: (cos(A+B), sin(A+B)) and (1, 0), as one set with (cos A, sin A) and (cos(-B), sin(-B)) the other set. [Graph both sets and show that their distances are equal].

Proof: $\sqrt{(\cos(A+B)-1)^2+(\sin(A+B)-0)^2} = \sqrt{(\cos A-\cos(-B))^2+(\sin A-\sin(-B))^2}$ and since both sides have a radical it can be dropped, also since cos(-B) = cos B but sin(-B) = - sin B from the unit circle, substituting gives:

$(\cos(A+B) - 1)^2 + \sin^2(A+B) = (\cos A - \cos B)^2 + (\sin A + \sin B)^2$ and squaring each parenthesis gives:

$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B) = \cos^2 A - 2\cos A \cos B + \cos^2 B + \sin^2 A + 2\sin A \sin B + \sin^2 B$ and by applying identity one three times we have: $2 - 2\cos(A+B) = 2 - 2\cos A \cos B + 2\sin A \sin B$. Subtracting 2 and then dividing every term by '-2' gives the identity: $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

c) **$\sin(x + y) = \sin x \cos y + \sin y \cos x$** called the sum formula for sines. This identity is established from identity 2 when it is realized that $\sin(x) = \cos(\pi/2 - x)$ and that $\cos(x) = \sin(\pi/2 - x)$. The proof goes as follows:

$\sin(x+y) = \cos\left(\frac{\pi}{2}-(x+y)\right) = \cos\left(\left(\frac{\pi}{2}-x\right)+(-y)\right) = \cos\left(\frac{\pi}{2}-x\right)\cos(-y)-\sin\left(\frac{\pi}{2}-x\right)\sin(-y)$, but each of these last four factors can be replaced with what they equal giving: sin(x+y) = sin x cos y + sin x sin y, which is the desired identity.

d) **$\cos(2x) = \cos^2 x - \sin^2 x$.** This identity comes from the second identity by letting every 'y' be replaced with 'x' and simplifying. Two other forms are obtained by using (a), solving first for $\cos^2 x$ and substituting, getting: **$\cos(2x) = 1 - 2\sin^2 x$** and then solving for $\sin^2 x$ and substituting, getting: **$\cos(2x) = 2\cos^2 x - 1$** .

e) **$\sin(2x) = 2 \sin x \cos x$** . This identity comes form the third identity by letting every 'y' be replaced with 'x' too.

f) **$\sin^2 x$** $= \dfrac{1-\cos(2x)}{2}$. This identity comes from the second form of d, by solving for $\sin^2 x$.

g) **$\cos^2 x$** $= \dfrac{1+\cos(2x)}{2}$. This identity comes from the third form of d, by solving for $\cos^2 x$.

2) Proofs and verifications.

a) Verifications are done by selecting any of the 17 angles from the Unit Circle and substituting them into any of the Magnificent Seven and showing that the results are equal:

Examples: let $x = \frac{2\pi}{3}$ or 120°, and $y = \frac{7\pi}{4}$ or 315°; therefore we verify identities 2 and 4 as follows:

$\cos(120° + 315°) = \cos \frac{2\pi}{3} \cos \frac{7\pi}{4} - \sin \frac{2\pi}{3} \sin \frac{7\pi}{4}$

$\cos(435°) = \mathbf{.259} = (-.5)(.707) - (.866)(-.707) = -.354 + .612 = \mathbf{.259}$

$\cos(2\frac{2\pi}{3}) = \cos^2 \frac{2\pi}{3} - \sin^2 \frac{2\pi}{3}$

$\cos\left(\frac{4\pi}{3}\right) = \mathbf{-.5} = (-.5)^2 - (.866)^2 = .25 - .75$

b) Proofs are done by taking an equation that may be an identity and using the Magnificent Seven to show that one member can be changed into the other member:

Examples:

$$\begin{aligned} 1 - 2\sin^2 x &= 2\cot(2x)\sin x \cos x \\ &= \cot(2x)\, 2\sin x \cos x \\ &= \frac{\cos(2x)}{\sin(2x)} \cdot \sin(2x) \\ &= \cos(2x) \\ 1 - 2\sin^2 x &= 1 - 2\sin^2 x \end{aligned}$$

$$\begin{aligned} \cos^4 x - \sin^4 x &= \cos(2x) \\ (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x) &= \\ \cos(2x) \cdot 1 &= \\ \cos(2x) &= \cos(2x) \end{aligned}$$