Math 111 – Lecture II: Solving Trig equations, Solving any triangle, Complex Numbers and Vectors

A) Solving equations that have trig functions demands that **single** trig functions are **isolated and solved for**, then the **angles** can be evaluated **using the Inverse function rules** plus the unit circle and/or function graphs. Essentially three types of equations will be encountered:

1) Simple equations that already have only one trig function.

Example: 6 – 3 sec x = 2; solving for sec x gives:

sec x = 1.333, therefore taking the inverse sec of both sides gives

$x = \sec^{-1}(1.333) = \cos^{-1}(.75) = 41.41°$, but since there are infinite answers a 4th quadrant answer is needed like -41.41° and adding 360°k to each gives: 41.41° + 360°k and -41.41° + 360°k.

Of course radian answers are just as good if not better sometimes so: $x = \begin{cases} .723 + 2\pi k \\ -.723 + 2\pi k \end{cases}$

2) Equations that need identities and/or algebra (especially factoring) to isolate single trig functions.

Example: 2 cos(2x) – sin x = -1; since there are two functions we use identity 4 with the sin x form:

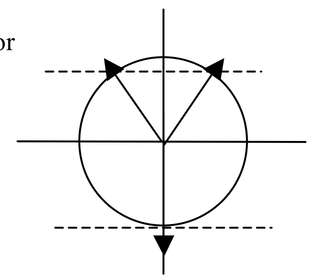
$2(1 - 2\sin^2 x) - \sin x = -1$, and now use algebra to get '0' on one side and write in ascending order

$2 - 4\sin^2 x - \sin x + 1 = 0$ or $3 - \sin x - 4\sin^2 x = 0$, the left side now factors giving:

(3 – 4 sin x) (1 + sin x) = 0. Setting each factor to zero and solving for sin x gives:

$\sin x = \frac{3}{4}$ and $\sin x = -1$ taking the $\sin^{-1}$ of both sides of each gives:

$x = \sin^{-1}(.75) = .85$ and $x = \sin^{-1}(-1) = -\frac{\pi}{2}$, but since there are infinite answers for both we refer to the unit circle figure and draw horizontal lines (the dotted lines) at y = .85 and y = -1 since the function was sin x, identify the extra angle and add



2πk to all three in this case: $x = \begin{cases} .85 + 2\pi k \\ 2.29 + 2\pi k \\ -\frac{\pi}{2} + 2\pi k \end{cases}$ (2.29 is π - .85)

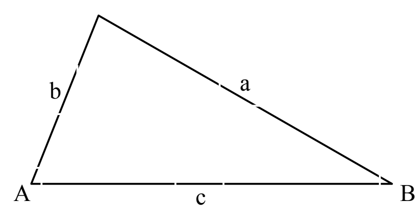
3) Equations that have a number with the 'x' inside a trig function, like tan(3x): the process requires isolating the trig function, using the inverse, getting infinite solutions and last of all dividing by '3' or multiplying by 1/3.

Example: 4 tan(3x) – 6 = 5; isolating tan gives: $\tan(3x) = \frac{11}{4}$, taking the inverse tan of both sides gives:

$3x = \tan^{-1}(2.75) = 1.22$, and the infinite solutions for tangent are: $3x = 1.22 + \pi k$, last we multiply both sides by 1/3:

$$\frac{1}{3}3x = x = \frac{1}{3}(1.22 + \pi k) = .407 + \frac{\pi}{3}k$$

B) Now we can solve any triangle, it does not have to be right, but we must know three of six things (the six are 3 angles and 3 sides). All appropriate sets of three can be put into the following 4 groups**: SAS, SSS, ASA and SSA**.

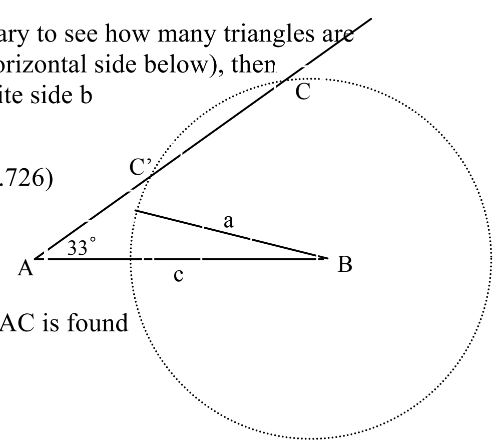


1) **The Law of Sines** handles the last two groups: **ASA** and the **ambiguous** case **SSA** since each of these gives us **an alphabet pair**. The ambiguous case can result in two triangles, one triangle or no triangle. The statement of the law is:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \textbf{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Example: If A = 33°, a = 6, and c = 8 find the other two angles and side b.

The three known values give SSA, the ambiguous case so a diagram is necessary to see how many triangles are possible for the two sides and angle. Draw the angle first, identify side c (the horizontal side below), then at the other end of c construct a circle of radius a. Notice it intersects the opposite side b twice, therefore there are two triangles possible for these measures.

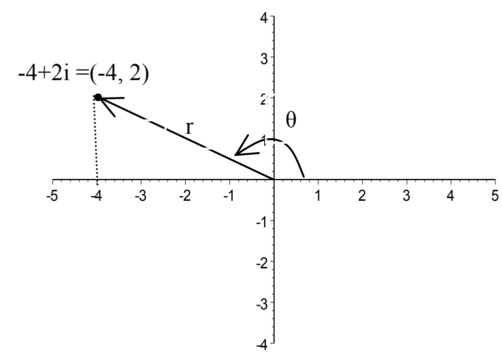


Now using the Law of sines: $\frac{\sin 33°}{6} = \frac{\sin C}{8}$ so sin C = .726, and $C = \sin^{-1}(.726)$ = 46.6°, but this is an acute angle so it must be the angle in the larger triangle (where the radius 'a' intersects the opposite side at the point C). The angle, AC'B would be the supplementary angle to C or, 133.4°. The angle CBA is 180°-33°-46.6°=100.4° and the angle C'BA is 180°-33°-133.4°=13.6°. The side AC is found using the Law of Sines: $\frac{\mathbf{AC}}{\mathbf{\sin 100.4}} = \frac{\mathbf{6}}{\mathbf{\sin 33}}$, so AC = 10.8 and similarly AC' is found

$\frac{\mathbf{AC'}}{\mathbf{\sin 13.6}} = \frac{\mathbf{6}}{\mathbf{\sin 33}}$, or AC' = 2.6. Therefore for the two possible triangles ABC' and ABC, The corresponding facts are: triangle ABC' has side b = 2.6, angle B = 13.6° and angle C' = 133.4°; triangle ABC has side b = 10.8, angle B = 100.4° and angle C = 46.6°.

2) **The Law of Cosines** handles the other two groups: **SAS and SSS.** The formula for the Law of Cosines can be stated three ways: $\mathbf{c^2 = a^2 + b^2 - 2ab\cos C}$ or $\mathbf{a^2 = b^2 + c^2 - 2bc\cos A}$ or $\mathbf{b^2 = a^2 + c^2 - 2ac\cos B}$
Example 1: If a = 4, b = 8, and c = 5, what are the measures of the three angles? Obviously this is the **SSS** case.
Solution: Since b is the largest side we use the last formula having cos of B and get: 64 = 16 + 25 – 40 cos B, solving for cos B gives, $\mathbf{\cos B = \frac{16+25-64}{40} = \frac{-23}{40} = -.575}$ so B = $\cos^{-1}(-.575)$ = 125.1° (notice it is an **obtuse angle**, $\cos^{-1}$ of a negative always is bigger than 90°). Now using the Law of Sines gives, $\frac{\mathbf{\sin A}}{\mathbf{5}} = \frac{\mathbf{\sin 125.1}}{\mathbf{8}}$, so sin A = .511 and therefore, A = $\sin^{-1}(.511)$ = 30.8° and since the angles of a triangle add up to 180°, angle C = 180°-125.1°-30.8° = 24.1°.
Note: Whenever you are able it is wise to **use the Law of Cosines to obtain the largest angle** of the triangle because it can give you an **obtuse angle**, the **Law of Sines never gives an obtuse angle**, why?
Example 2: If a = 10, b = 9, and C = 68°, what are the other two angles and the unknown side?
Solution: Since no alphbet pair is given, the case is **SAS** and since angle C is known we will use the first formula: c^2 = 100 + 81 – 180 cos 68° = 181 – 67.43 =113.57, so c = 10.66 and since this is larger than the other two sides the Law of sines can be used to find the two angles: $\frac{\mathbf{\sin A}}{\mathbf{10}} = \frac{\mathbf{\sin 68}}{\mathbf{10.66}}$, so sin A = .87 and A = 60.4°, therefore B = 51.6°.

C) Complex Numbers and trigonometry (PolarC can be used on the calculator)



1) Trigonometry form:
   i. Complex numbers previously have been defined two ways: In **term form a + bi**, and in **ordered pair form (a, b)**. For example the complex number -4 + 2i or (-4, 2) on the graph,
   ii. Now if a line is drawn from the point (0, 0) to the point (-4, 2), it would have an angle θ from the positive x-axis and it would be 'r' units long. It would now be possible to represent the ordered pair (-4, 2) using r and θ: r cos θ for the x, and r sin θ for the y giving us the **trig form: r cos θ + i·r sin θ**.
   Therefore, for the point (-4, 2),
   $r = \sqrt{(-4)^2 + 2^2} = \sqrt{16+4} = \sqrt{20}$ = 4.47 and $\tan\theta = \frac{2}{-4}$ so
   θ = $\tan^{-1}(-.5)$ = -26.565° but it can be seen that the correct answer is in the 2nd quadrant or -26.565° + 180° which is equal to 153.435°. That means the trig form of the complex number (-4, 2) is
   4.47 cos 153.44° + i·4.47 sin 153.44°
   c) We do shorten the trig form from **r cos θ + i·r sin θ** to **r cis θ** so the trig form for (-4, 2) is 4.47 cis 153.44°. It is also possible to find the 'r' and the 'θ' using the 'PolarC' mode on your calculator; change your mode to 'PolarC', then enter the ordered pair '(-4, 2)' and push enter and you get (4.47 ∠ 153.44°) giving you 'r' and 'θ'.

2) The advantages of trig form are in multiplication and division of complex numbers and taking powers:
   a) Multiplication: $\mathbf{(r_1 \text{ cis } \theta_1)(r_2 \text{ cis } \theta_2) = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)}$ Example: (3 cis 25°) (5 cis 75°) = 15 cis 100°
   For **homework** prove this rule by writing both complex numbers in long form, foiling, and using identities. Also for **homework** prove the next rule.
   b) Division: $\mathbf{\frac{r_1 \text{ cis } \theta_1}{r_2 \text{ cis } \theta_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)}$ Example: $\mathbf{\frac{4 \text{ cis } 30°}{2 \text{ cis } 80°} = 2 \text{ cis } (-50°)}$

3) DeMoivre's Theorem: powers and roots of complex numbers.

a) DeMoivre's Theorem states: $\mathbf{(r\ cis\ \theta)^n = r^n\ cis\ (n\theta)}$ Example: $(2\ cis\ 24^\circ)^3 = 8\ cis\ 72^\circ$

b) Taking roots of complex numbers: $\mathbf{\sqrt[n]{r\,cis\,\theta} = r^{\frac{1}{n}}\ cis\left(\frac{\theta}{n} + k\,\frac{360^\circ}{n}\right)}$, this means there will be n roots and k will take on the values, 0, 1, 2, …, (n-1).

Example 1: $\mathbf{\sqrt[4]{16\,cis\,200^\circ} = 16^{\frac{1}{4}}\ cis\left(\frac{200^\circ}{4} + k\,\frac{360^\circ}{4}\right) = 2\,cis(50^\circ + 90^\circ k)}$ and the four answers are: 2 cis 50˚, 2 cis 140˚, 2 cis 230˚, and 2 cis 320˚.

Example 2: $\mathbf{\sqrt[3]{-8i} = \sqrt[3]{8\,cis\,270^\circ} = \sqrt[3]{8}cis\left(\frac{270^\circ}{3} + k\,\frac{360^\circ}{3}\right) = 2\,cis\,(90^\circ + 120^\circ k)}$ or 2 cis 90˚, 2cis 210˚, & 2 cis 330˚

D) Vectors introduced through examples:

1) Vectors have magnitude (size) and direction (angle). For instance if the wind was blowing at 40 mph from the southwest, its magnitude would be 40 mph and the direction would be 45˚ East of North.

2) The following problem will illustrate the effects of vectors and how we will use them:
Suppose an airplane, P, was heading 35˚ West of North at 450 mph and suppose the wind, W, was blowing at 105˚ West of North at 150 mph, what direction would the plane actually be heading and what would its speed actually be?
Solution: The diagonal arrow, x, in the diagram would be the vector representing the plane, it is the diagonal of the parallelogram and the angle between the plane and the wind (105˚ - 35˚) is 70˚. This measure is not helpful, however, except in determining the angle in the top left hand corner of the parallelogram. That angle is supplementary to 70˚ and therefore it is 110˚. The two things we need to find are x and θ and since we know SAS, the Law of Cosines can be used:
$x^2 = 450^2 + 150^2 - 900 \cdot 150 \cos 110^\circ = 271172.72$, so x = 520.7 mph and θ is found using the Law of Sines: $\frac{\sin \boldsymbol{\theta}}{\mathbf{150}} = \frac{\sin \mathbf{110}}{\mathbf{520.7}}$, so sin θ = .271 and θ = 15.7˚.
Therefore, the plane is actually traveling at 520.7 mph, and it is heading 15.7˚ + 35˚ = 50.7˚ degrees West of North.